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On certain new formulas for the Horn's hypergeometric functions $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$

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Abstract
Inspired by the recent work Sahin and Agha gave recursion formulas for $\mathcal{G}_1$ and $\mathcal{G}_2$ Horn hypergeometric functions (Sahin and Agha in Miskolc Math Notes 16(2):1153–1162, 2015). The object of work is to establish several new recursion relations, relevant differential recursion formulas, new integral operators, infinite summations and interesting results for Horn's hypergeometric functions $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$.

Keywords Contiguous relations · Infinite summation formulas · Recursion formulas · Horn functions

Mathematics Subject Classification Primary 33C05 · 33C20; Secondary 33C15 · 11J72

1 Introduction and notations

The problem of recursion formulas is famous in the field of hypergeometric functions with respect to their parameters, see example [2, 4, 5, 7, 8, 11, 15, 18]. In this direction, Sahin [12], Sahin and Agha [13] studied the recursion formulas of $\mathcal{G}_1$ and $\mathcal{G}_2$ Horn hypergeometric functions. Pathan et al. [9], Shehata and Moustafa [14] have investigated the properties and numerous extensions of various recursion formulas of Horn hypergeometric functions in depth.

In [10, 16, 17], the $\Gamma(\rho)$ Gamma function is defined as

$$\Gamma(\rho) = \int_0^\infty e^{-t} t^{\rho-1} dt, \quad \Re(\rho) > 0. \tag{1.1}$$

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The Pochhammer symbol $(\rho)_m$ is given by

$$(\rho)_m = \frac{\Gamma(\rho+m)}{\Gamma(\rho)} = \begin{cases} 1, & (\rho \in \mathbb{C}, m=0); \\ \rho(\rho+1)(\rho+2)\ldots(\rho+m-1), & (\rho \in \mathbb{C}, m \in \mathbb{N}), \end{cases} \quad (1.2)$$

$$(\rho)_{-\ell} = \frac{(-1)^\ell}{(1-\rho)_\ell}, \quad (\rho \neq 0, \pm1, \pm2, \pm3, \ldots, \ell \in \mathbb{N}), \quad (1.3)$$

$$(\rho)_{m-\ell} = \begin{cases} 0, & \ell > m; \\ \frac{(-1)^\ell (\rho)_m}{(1-m-\rho)_\ell}, & 0 \leq \ell \leq m, m, \ell \in \mathbb{N} \end{cases} \quad (1.4)$$

and

$$(\rho)_{2m-\ell} = \begin{cases} 0, & \ell > 2m; \\ \frac{(-1)^\ell (\rho)_{2m}}{(1-2m-\rho)_\ell}, & 0 \leq \ell \leq 2m, m, \ell \in \mathbb{N}, \end{cases} \quad (1.5)$$

where $\mathbb{N}$ and $\mathbb{C}$ represent the set of complex and natural numbers.

Following abbreviated notations to keep the paper are used (see [10, 16, 17]). We write

$$\begin{aligned} (\rho)_{m+1} &= \rho(\rho+1)_m = (\rho+m)(\rho)_m, \\ (\rho)_{m-1} &= \frac{1}{\rho-1}(\rho-1)_m; \quad \rho \neq 1, \\ (\rho+1)_m &= \left(1+\frac{m}{\rho}\right)(\rho)_m; \quad \rho \neq 0, \\ (\rho-1)_m &= (\rho-1)(\rho)_{m-1} = \frac{\rho-1}{\rho+m-1}(\rho)_m; \quad \rho \neq 1-m. \end{aligned} \quad (1.6)$$

The Horn's functions $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are defined by (see [6, 16, 17])

$$\begin{aligned} \mathcal{G}_1 &= \mathcal{G}_1(\alpha, \beta, \gamma; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{n-m}(\gamma)_{m-n}}{m!n!} x^m y^n, \\ &(\beta, \gamma \text{ satisfy conditions (1.3) and (1.4)}, |x| < r, |y| < s, r+s < 1), \end{aligned} \quad (1.7)$$

$$\begin{aligned} \mathcal{G}_2 &= \mathcal{G}_2(\alpha, \beta, \gamma, \delta; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_m(\beta)_n(\gamma)_{n-m}(\delta)_{m-n}}{m!n!} x^m y^n, \\ &(\gamma, \delta \text{ satisfy conditions (1.3) and (1.4)}, |x| < 1, |y| < 1) \end{aligned} \quad (1.8)$$

and

$$\begin{aligned} \mathcal{G}_3 &= \mathcal{G}_3(\alpha, \beta; x, y) = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{2n-m}(\beta)_{2m-n}}{m!n!} x^m y^n, \\ &(\alpha, \beta \text{ satisfy conditions (1.3) and} \\ &\text{(1.5)}, |x| < r, |y| < s, 27r^2s^2 + 18rs \pm 4(r-s) = 1), \end{aligned} \quad (1.9)$$

In order to keep the paper completed, the investigation of recursion formulas of Horn functions $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are important, the most famous interesting and neglect any further discussions [3, 9, 14].

Motivated actually by the verified workable for applications of Horn functions $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ in many numerous areas of mathematics, physical, engineering and statistical sciences. This paper is complementary to publish paper [13], is prepared as follows. We establish new recursion formulas for the $\mathcal{G}_3(\alpha \pm \ell, \beta; x, y)$ to $\ell \in \mathbb{N}$ in Sect. 2. We present new differential recursion formulas for the $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ in Sect. 3. In Sect. 4, we discuss the various new integral operators of the $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$. In Sect. 5, we derive the various infinite summation formulas of the $\mathcal{G}_1, \mathcal{G}_2$ and $\mathcal{G}_3$. Section 6, several concluding remarks of outcomes are described.

2 Recursion formulas of Horn function $\mathcal{G}_3$

Some recursion formulas of Horn function $\mathcal{G}_3$ are given here.

Theorem 2.1 *For $\ell \in \mathbb{N}$. Recursion formulas for Horn function $\mathcal{G}_3$ are as follows*

$$\begin{aligned}\mathcal{G}_3(\alpha+\ell, \beta; x, y) &= \mathcal{G}_3 + \frac{2y}{\beta-1}\sum_{s=1}^{\ell}(\alpha+s)\mathcal{G}_3(\alpha+s+1, \beta-1; x, y)\\ &-\beta(\beta+1)x\sum_{s=1}^{\ell}\frac{1}{(\alpha+s-1)(\alpha+s-2)}\mathcal{G}_3(\alpha+s-2, \beta+2; x, y),\\ &\beta \neq 1, \alpha \neq 1-s, \alpha \neq 2-s, s \in \mathbb{N}\end{aligned} \tag{2.1}$$

and

$$\begin{aligned}\mathcal{G}_3(\alpha, \beta+\ell; x, y) &= \mathcal{G}_3 + \frac{2x}{\alpha-1}\sum_{s=1}^{\ell}(\beta+s)\mathcal{G}_3(\alpha-1, \beta+s+1; x, y)\\ &-\alpha(\alpha+1)y\sum_{s=1}^{\ell}\frac{1}{(\beta+s-1)(\beta+s-2)}\mathcal{G}_3(\alpha+2, \beta+s-2; x, y),\\ &\alpha \neq 1, \beta \neq 1-s, \beta \neq 2-s, s \in \mathbb{N}.\end{aligned} \tag{2.2}$$

Proof From (1.9) and transformation

$$(\alpha+1)_{2n-m} = (\alpha)_{2n-m}\left(1+\frac{2n-m}{\alpha}\right), \quad \alpha \neq 0, \tag{2.3}$$

we get the relation:

$$\begin{aligned}\mathcal{G}_3(\alpha+1, \beta; x, y) &= \mathcal{G}_3 + \frac{2(\alpha+1)y}{\beta-1}\mathcal{G}_3(\alpha+2, \beta-1; x, y)\\ &-\frac{\beta(\beta+1)x}{\alpha(\alpha-1)}\mathcal{G}_3(\alpha-1, \beta+2; x, y), \quad \beta \neq 1, \alpha \neq 0, \alpha \neq 1.\end{aligned} \tag{2.4}$$

By applying this contiguous relation to Horn function $\mathcal{G}_3$ with the parameter $\alpha = \alpha + \ell$ through relation (2.4) for ℓ times, we get (2.1).

Using (1.9) and the relation

$$(\beta+1)_{2m-n} = (\beta)_{2m-n}\left(1+\frac{2m-n}{\beta}\right), \beta \neq 0, \tag{2.5}$$

we obtain the contiguous function

$$\begin{aligned}\mathcal{G}_3(\alpha, \beta+1; x, y) &= \mathcal{G}_3 + \frac{2(\beta+1)x}{\alpha-1}\mathcal{G}_3(\alpha-1, \beta+2; x, y)\\ &-\frac{a(\alpha+1)y}{\beta(\beta-1)}\mathcal{G}_3(\alpha+2, \beta-1; x, y), \quad \alpha \neq 1, \beta \neq 0, 1.\end{aligned} \tag{2.6}$$

By iterating this method on $\mathcal{G}_3$ with the parameter $\beta+\ell$ for ℓ times in (2.6), we obtain the relation (2.2). □

3 Differential recursion formulas of $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$

Here, we obtain several differential recursion formulas of Horn's functions $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ through using differential operators $\Theta_x = x\frac{\partial}{\partial x}$ and $\Theta_y = y\frac{\partial}{\partial y}$.

Theorem 3.1 *Differential recursion formulas of function $\mathcal{G}_3$ are as follows*

$$\mathcal{G}_3(\alpha+\ell,\beta;x,y) = \prod_{i=0}^{\ell-1}\left(1+\frac{2\Theta_y-\Theta_x}{\alpha+i}\right)\mathcal{G}_3(\alpha,\beta;x,y), \quad \alpha \neq 0 \tag{3.1}$$

and

$$\mathcal{G}_3(\alpha,\beta+\ell,\gamma;x,y) = \prod_{i=0}^{\ell-1}\left(1+\frac{2\Theta_x-\Theta_y}{\beta+i}\right)\mathcal{G}_3(\alpha,\beta;x,y), \quad \beta \neq 0. \tag{3.2}$$

Proof Defining the differential operators

$$\begin{aligned}\Theta_x x^m &= x\frac{\partial}{\partial x}x^m = mx^m,\\ \Theta_y y^n &= y\frac{\partial}{\partial y}y^n = ny^n.\end{aligned} \tag{3.3}$$

By using the above differential operators and (2.3), we get the differential recursion formula for $\mathcal{G}_3$

$$\begin{aligned}\mathcal{G}_3(\alpha+1,\beta;x,y) &= \sum_{m,n=0}^{\infty}\frac{\left(1+\frac{2n-m}{\alpha}\right)(\alpha)_{2n-m}(\beta)_{2m-n}}{m!n!}x^m y^n\\ &= \mathcal{G}_3(\alpha,\beta;x,y) + 2\frac{\Theta_y}{\alpha}\mathcal{G}_3(\alpha,\beta;x,y) - \frac{\Theta_x}{\alpha}\mathcal{G}_3(\alpha,\beta;x,y).\end{aligned} \tag{3.4}$$

By iterating this method on $\mathcal{G}_3$ for ℓ times in (3.4), we obtain (3.1). Similarly, using the above differential operators (3.3) and iterating this technique on $\mathcal{G}_3$ for ℓ times, we obtain (3.2). □

Theorem 3.2 *Differential recursion formulas for the functions $\mathcal{G}_1$ and $\mathcal{G}_2$*

$$\begin{aligned}\mathcal{G}_1(\alpha+\ell,\beta,\gamma;x,y) &= \prod_{i=0}^{\ell-1}\left(1+\frac{\Theta_x+\Theta_y}{\alpha+i}\right)\mathcal{G}_1(\alpha,\beta,\gamma;x,y), \quad \alpha\neq 0,\\ \mathcal{G}_2(\alpha+\ell,\beta,\gamma,\delta;x,y) &= \prod_{i=0}^{\ell-1}\left(1+\frac{\Theta_x}{\alpha+i}\right)\mathcal{G}_2(\alpha,\beta,\gamma,\delta;x,y), \quad \alpha\neq 0,\end{aligned} \tag{3.5}$$

$$\begin{aligned}\mathcal{G}_1(\alpha,\beta+\ell,\gamma;x,y) &= \prod_{i=0}^{\ell-1}\left(1+\frac{\Theta_y-\Theta_x}{\beta+i}\right)\mathcal{G}_1(\alpha,\beta,\gamma;x,y), \quad \beta\neq -i,\\ \mathcal{G}_2(\alpha,\beta+\ell,\gamma,\delta;x,y) &= \prod_{i=0}^{\ell-1}\left(1+\frac{\Theta_y}{\beta+i}\right)\mathcal{G}_2(\alpha,\beta,\gamma,\delta;x,y), \quad \beta\neq -i,\end{aligned} \tag{3.6}$$

$$\mathcal{G}_1(\alpha,\beta,\gamma+\ell;x,y)=\prod_{i=0}^{\ell-1}\left(1+\frac{\Theta_x-\Theta_y}{\gamma+i}\right)\mathcal{G}_1(\alpha,\beta,\gamma;x,y),\quad \gamma\neq -i,$$

$$\mathcal{G}_2(\alpha,\beta,\gamma+\ell,\delta;x,y)=\prod_{i=0}^{\ell-1}\left(1+\frac{\Theta_y-\Theta_x}{\gamma+i}\right)\mathcal{G}_2(\alpha,\beta,\gamma,\delta;x,y),\quad \gamma\neq i \tag{3.7}$$

and

$$\mathcal{G}_2(\alpha,\beta,\gamma,\delta+\ell;x,y)=\prod_{i=0}^{\ell-1}\left(1+\frac{\Theta_x-\Theta_y}{\delta+i}\right)\mathcal{G}_2(\alpha,\beta,\gamma,\delta;x,y),\quad \delta\neq -i \tag{3.8}$$

are established.

Theorem 3.3 *The derivative formulas for the functions $\mathcal{G}_3$, $\mathcal{G}_1$ and $\mathcal{G}_2$*

$$\frac{\partial^\ell}{\partial x^\ell}\mathcal{G}_3(\alpha,\beta;x,y)=\frac{(-1)^\ell(\beta)_{2\ell}}{(1-\alpha)_\ell}\mathcal{G}_3(\alpha-\ell,\beta+2\ell;x,y),\quad \alpha\neq 1,2,3,\ldots \tag{3.9}$$

$$\frac{\partial^\ell}{\partial y^\ell}\mathcal{G}_3(\alpha,\beta;x,y)=\frac{(-1)^\ell(\alpha)_{2\ell}}{(1-\beta)_\ell}\mathcal{G}_3(\alpha+2\ell,\beta-\ell;x,y),\quad \beta\neq 1,2,3,\ldots \tag{3.10}$$

$$\frac{\partial^\ell}{\partial x^\ell}\mathcal{G}_1=\frac{(-1)^\ell(\alpha)_\ell}{(1-\beta)_\ell(\gamma)_\ell}\mathcal{G}_1(\alpha+\ell,\beta-\ell,\gamma+\ell;x,y),\quad \beta\neq 1,2,3,\ldots,$$
$$\frac{\partial^\ell}{\partial x^\ell}\mathcal{G}_2=\frac{(-1)^\ell(\alpha)_\ell}{(1-\gamma)_\ell(\delta)_\ell}\mathcal{G}_2(\alpha+\ell,\beta,\gamma-\ell,\delta+\ell;x,y),\quad \gamma\neq 1,2,3,\ldots \tag{3.11}$$

and

$$\frac{\partial^\ell}{\partial y^\ell}\mathcal{G}_1=\frac{(-1)^\ell(\alpha)_\ell(\beta)_\ell}{(1-\gamma)_\ell}\mathcal{G}_1(\alpha+\ell,\beta+\ell,\gamma-\ell;x,y),\quad \gamma\neq 1,2,3,\ldots,$$
$$\frac{\partial^\ell}{\partial y^\ell}\mathcal{G}_2=\frac{(-1)^\ell(\beta)_\ell(\gamma)_\ell}{(1-\delta)_\ell}\mathcal{G}_2(\alpha,\beta+\ell,\gamma+\ell,\delta-\ell;x,y),\quad \delta\neq 1,2,3,\ldots \tag{3.12}$$

are derived.

Proof Differentiating (1.9) with respect to x, we get

$$\frac{\partial}{\partial x}\mathcal{G}_3(\alpha,\beta;x,y)=\sum_{m,n=0}^{\infty}\frac{m(\alpha)_{2n-m}(\beta)_{2m-n}}{m!n!}x^{m-1}y^n=\sum_{m,n=0}^{\infty}\frac{(\alpha)_{2n-m-1}(\beta)_{2m-n+2}}{m!n!}x^my^n$$
$$=\frac{\beta(\beta+1)}{\alpha-1}\sum_{m,n=0}^{\infty}\frac{(\alpha-1)_{2n-m}(\beta+2)_{2m-n}}{m!n!}x^my^n=\frac{\beta(\beta+1)}{\alpha-1}\mathcal{G}_3(\alpha-1,\beta+2;x,y).$$

Repeating the above relation, we arrive at

$$\begin{aligned}\frac{\partial^\ell}{\partial x^\ell}\mathcal{G}_3 &= \frac{\beta(\beta+1)\ldots(\beta+2\ell-1)}{(\alpha-1)(\alpha-2)\ldots(\alpha-\ell)}\mathcal{G}_3(\alpha-\ell,\beta+2\ell;x,y)\\ &= \frac{(-1)^\ell\beta(\beta+1)\ldots(\beta+2\ell-1)}{(1-\alpha)(2-\alpha)\ldots(\ell-\alpha)}\mathcal{G}_3(\alpha-\ell,\beta+2\ell;x,y)\\ &= \frac{(-1)^\ell(\beta)_{2\ell}}{(1-\beta)_\ell}\mathcal{G}_3(\alpha-\ell,\beta+2\ell;x,y).\end{aligned}$$

The derivative (1.9) with respect to y, we proceed

$$\frac{\partial}{\partial y}\mathcal{G}_3(\alpha,\beta;x,y) = \frac{\alpha(\alpha+1)}{\beta-1}\mathcal{G}_3(\alpha+2,\beta-1;x,y).$$

By iterating this approach on the rth derivatives, we get (3.10). Similar way, we prove the Eqs. (3.11) and (3.12). □

4 Integral operators of $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$

Now, we define an integral operator $\check{\mathbb{I}}$ acting on the Horn functions $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ such that [1]

$$\check{\mathbb{I}} = \frac{1}{x}\int_0^x dx + \frac{1}{y}\int_0^y dy, \tag{4.1}$$

where the integration is carried out with respect to each variable individually on the assumption that the other is constant.

Theorem 4.1 *The following integration formulas of the Horn functions $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ hold true:*

$$\begin{aligned}\check{\mathbb{I}}^2\mathcal{G}_1 &= \frac{\beta(\beta+1)}{x^2(\alpha-1)(\alpha-2)(\gamma-1)(\gamma-2)}\mathcal{G}_1(\alpha-2,\beta+2,\gamma-2;x,y)\\ &+\frac{2}{xy(\alpha-1)(\alpha-2)}\mathcal{G}_1(\alpha-2,\beta,\gamma;x,y)\\ &+\frac{\gamma(\gamma+1)}{y^2(\alpha-1)(\alpha-2)(\beta-1)(\beta-2)}\mathcal{G}_1\\ &\quad(\alpha-2,\beta-2,\beta+2;x,y),\ \alpha,\beta,\gamma\neq 1,2,\ x,y\neq 0,\end{aligned}\tag{4.2}$$

$$\begin{aligned}\check{\mathbb{I}}^2\mathcal{G}_2 &= \frac{\gamma(\gamma+1)}{x^2(\alpha-1)(\alpha-2)(\delta-1)(\delta-2)}\mathcal{G}_2(\alpha-2,\beta,\gamma+2,\delta-2;x,y)\\ &+\frac{2}{xy(\alpha-1)(\alpha-2)(\beta-1)(\beta-2)}\mathcal{G}_2(\alpha-2,\beta-2,\gamma,\delta;x,y)\\ &+\frac{\delta(\delta+1)}{y^2(\beta-1)(\beta-2)(\gamma-1)(\gamma-2)}\mathcal{G}_2\\ &\quad(\alpha,\beta-2,\gamma-2,\delta+2;x,y),\ \alpha,\beta,\gamma\neq 1,2,\ x,y\neq 0\end{aligned}\tag{4.3}$$

and

$$\begin{aligned}\check{\mathbb{I}}^2\mathcal{G}_3 &= \frac{a(\alpha+1)}{x^2(\beta-1)(\beta-2)(\beta-3)(\beta-4)}\mathcal{G}_3(\alpha+2,\beta-4;x,y)\\ &+\frac{2}{xy(\alpha-1)(\beta-1)}\mathcal{G}_3(\alpha-1,\beta-1;x,y)\\ &+\frac{\beta(\beta+1)}{y^2(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}\mathcal{G}_3(\alpha-4,\beta+2;x,y),\ \alpha,\beta\neq 1,2,3,4,x,y\neq 0.\end{aligned} \tag{4.4}$$

Proof Acting the definition $\check{\mathbb{I}}$ on the function $\mathcal{G}_1$, we have

$$\begin{aligned}\check{\mathbb{I}}\mathcal{G}_1 &= \sum_{m,n=0}^{\infty}\left(\frac{1}{m+1}+\frac{1}{n+1}\right)\frac{(\alpha)_{m+n}(\beta)_{n-m}(\gamma)_{m-n}}{m!n!}x^m y^n\\ &= \frac{\beta}{x(\alpha-1)(\gamma-1)}\mathcal{G}_1(\alpha-1,\beta+1;\gamma-1;x,y)\\ &\quad+\frac{\gamma}{y(\alpha-1)(\beta-1)}\mathcal{G}_1(\alpha-1,\beta-1,\gamma+1;x,y),\quad \alpha,\beta,\gamma\neq 1,x,y\neq 0.\end{aligned}$$

We can be written $\check{\mathbb{I}}=\check{\mathbb{I}}_x+\check{\mathbb{I}}_y$ where $\check{\mathbb{I}}_x=\frac{1}{x}\int_0^x dx$ and $\check{\mathbb{I}}_y=\frac{1}{y}\int_0^y dy$, then the operator $\check{\mathbb{I}}^2$ is such that

$$\begin{aligned}\check{\mathbb{I}}^2 &= \check{\mathbb{I}}\check{\mathbb{I}} = (\check{\mathbb{I}}_x)^2+2\check{\mathbb{I}}_x\check{\mathbb{I}}_y+(\check{\mathbb{I}}_x)^2\\ &= \frac{1}{x^2}\int_0^x\int_0^x dxdx+\frac{2}{xy}\int_0^y\int_0^x dxdy+\frac{1}{y^2}\int_0^y\int_0^y dydy.\end{aligned}$$

Using the integral operator $\check{\mathbb{I}}^2$, we get (4.2). Similarly, we obtain the relations (4.3) and (4.4).

Theorem 4.2 *The new integration formulas of the Horn functions $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ hold true:*

$$\begin{aligned}\check{\mathbb{I}}^{\ell}\mathcal{G}_1 &= \frac{1}{x^{\ell}y^{\ell}(1-\alpha)_{2\ell}}\prod_{k=1}^{\ell}(\Theta_x+\Theta_y-k+1)\mathcal{G}_1\\ &\quad(\alpha-2\ell,\beta,\gamma;x,y),\quad \alpha\neq 1,2,3,\ldots,x,y\neq 0,\end{aligned} \tag{4.5}$$

$$\begin{aligned}\check{\mathbb{I}}^{\ell}\mathcal{G}_2 &= \frac{1}{x^{\ell}y^{\ell}(1-\alpha)_{2\ell}(1-\beta)_{\ell}}\\ &\quad\times\prod_{k=1}^{\ell}(\Theta_x+\Theta_y-k+1)\mathcal{G}_2(\alpha-\ell,\beta-\ell,\gamma,\delta;x,y),\quad \alpha,\beta\neq 1,2,3,\ldots,x,y\neq 0\end{aligned} \tag{4.6}$$

and

$$\begin{aligned}\check{\mathbb{I}}^{\ell}\mathcal{G}_3 &= \frac{1}{x^{\ell}y^{\ell}(1-\alpha)_{\ell}(1-\beta)_{\ell}}\prod_{k=1}^{\ell}(\Theta_x+\Theta_y-k+1)\mathcal{G}_3\\ &\quad(\alpha-\ell,\beta-\ell;x,y),\quad \alpha,\beta\neq 1,2,3,\ldots,x,y\neq 0.\end{aligned} \tag{4.7}$$

Proof Using the operators $\check{\mathbb{I}}$, Θ_x and Θ_y, we get

$$\check{\mathbb{I}}\mathcal{G}_1 = \sum_{m,n=1}^{\infty}\frac{(m+n+2)(\alpha)_{m+n}(\beta)_{n-m}(\gamma)_{m-n}}{(m+1)!(n+1)!}x^m y^n$$

$$= \sum_{m,n=0}^{\infty} \frac{(m+n)(\alpha)_{m+n-2}(\beta)_{n-m}(\gamma)_{m-n}}{m!n!} x^{m-1} y^{n-1}$$
$$= \frac{\Theta_x + \Theta_y}{xy(\alpha-1)(\alpha-2)} \mathcal{G}_1(\alpha-2, \beta, \gamma; x, y), \quad \alpha \neq 1, 2, x, y \neq 0.$$

Iterating, for ℓ times, this integral $\check{\mathbb{I}}$ and differential operators on $\mathcal{G}_1$, we get the formula (4.5). Similar, we obtain the formulas (4.6) and (4.7). □

Theorem 4.3 *For Horn functions $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$, we have the integral operators $\check{\mathbb{I}}_x^{\ell}$ and $\check{\mathbb{I}}_y^{\ell}$:*

$$\check{\mathbb{I}}_x^{\ell}\, \mathcal{G}_1 = \frac{(\beta)_\ell}{x^\ell (1-\alpha)_\ell (1-\gamma)_\ell} \mathcal{G}_1(\alpha-\ell, \beta+\ell, \gamma-\ell; x, y), \quad \alpha, \gamma \neq 1, 2, 3, \ldots, x \neq 0, \tag{4.8}$$

$$\check{\mathbb{I}}_y^{\ell}\, \mathcal{G}_1 = \frac{(-1)^\ell (\gamma)_\ell}{y^\ell (1-\alpha)_\ell (1-\beta)_\ell} \mathcal{G}_1(\alpha-\ell, \beta-\ell, \gamma+\ell; x, y), \quad \alpha, \beta \neq 1, 2, 3, \ldots, y \neq 0,$$
$$\left(\check{\mathbb{I}}_x \check{\mathbb{I}}_y\right)^\ell \mathcal{G}_1 = \frac{(-1)^\ell}{x^\ell y^\ell (1-\alpha)_\ell} \mathcal{G}_1(\alpha-\ell, \beta, \gamma; x, y), \quad \alpha \neq 1, 2, 3, \ldots, x, y \neq 0, \tag{4.9}$$

$$\check{\mathbb{I}}_x^{\ell}\, \mathcal{G}_2 = \frac{(\gamma)_\ell}{x^\ell (1-\alpha)_\ell (1-\delta)_\ell} \mathcal{G}_2(\alpha-\ell, \beta, \gamma+\ell, \delta-\ell; x, y), \quad \alpha, \delta \neq 1, 2, 3, \ldots, x \neq 0,$$
$$\check{\mathbb{I}}_y^{\ell}\, \mathcal{G}_2 = \frac{(-1)^\ell (\delta)_\ell}{y^\ell (1-\beta)_\ell (1-\gamma)_\ell} \mathcal{G}_2(\alpha, \beta-\ell, \gamma-\ell, \delta+\ell; x, y), \quad \beta, \gamma \neq 1, 2, 3, \ldots, y \neq 0,$$
$$\left(\check{\mathbb{I}}_x \check{\mathbb{I}}_y\right)^\ell \mathcal{G}_2 = \frac{1}{x^\ell y^\ell (1-\alpha)_\ell (1-\beta)_\ell} \mathcal{G}_2(\alpha-\ell, \beta-\ell, c, \delta; x, y), \quad \alpha, \beta \neq 1, 2, 3, \ldots, x, y \neq 0 \tag{4.10}$$

and

$$\check{\mathbb{I}}_x^{\ell}\, \mathcal{G}_3 = \frac{(-1)^\ell (\alpha)_\ell}{x^\ell (1-\beta)_{2\ell}} \mathcal{G}_3(\alpha+\ell, \beta-2\ell; x, y), \quad \beta \neq 1, 2, 3, \ldots, x \neq 0,$$
$$\check{\mathbb{I}}_y^{\ell}\, \mathcal{G}_3 = \frac{(-1)^\ell (\beta)_\ell}{y^\ell (1-\alpha)_{2\ell}} \mathcal{G}_3(\alpha-2\ell, \beta+\ell; x, y), \quad \alpha \neq 1, 2, 3, \ldots, y \neq 0,$$
$$\left(\check{\mathbb{I}}_x \check{\mathbb{I}}_y\right)^\ell \mathcal{G}_3 = \frac{1}{x^\ell y^\ell (1-\alpha)^\ell (1-\beta)_\ell} \mathcal{G}_3$$
$$(\alpha-\ell, \beta-\ell; x, y), \quad \alpha, \beta \neq 1, 2, 3, \ldots, x, y \neq 0. \tag{4.11}$$

Proof Using the operator related to $\check{\mathbb{I}}_x$, we get

$$\check{\mathbb{I}}_x\, \mathcal{G}_1 = \frac{\beta}{x(\alpha-1)(\gamma-1)} \mathcal{G}_1(\alpha-1, \beta+1, \gamma-1; x, y), \quad \alpha, \gamma \neq 1, x \neq 0.$$

By making use of the operation for ℓ times, we get the desired (4.8).

Similarly, for $\mathfrak{I}_y$ and $\mathfrak{I}_x \mathfrak{I}_y$, by applying use of the above operations, we obtain (4.9)–(4.11).

Now, the operator $\check{\mathbf{I}}$ [1] is considered in the form

$$\check{\mathbf{I}} = \int_0^x dx + \int_0^y dy. \tag{4.12}$$

Theorem 4.4 *The integral operators for $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are*

$$\begin{aligned}\breve{\mathbf{I}}^2\mathcal{G}_1 &= \frac{\beta(\beta+1)}{(\alpha-1)(\alpha-2)(\gamma-1)(\gamma-2)}\mathcal{G}_1(\alpha-2,\beta+2,\gamma-2;x,y)\\ &+\frac{2}{(\alpha-1)(\alpha-2)}\mathcal{G}_1(\alpha-2,\beta,\gamma;x,y)\\ &+\frac{\gamma(\gamma+1)}{(\alpha-1)(\alpha-2)(\beta-1)(\beta-2)}\mathcal{G}_1(\alpha-2,\beta-2,\gamma+2;x,y),\quad \alpha,\beta,\gamma\neq 1,2,\end{aligned} \tag{4.13}$$

$$\begin{aligned}\breve{\mathbf{I}}^2\mathcal{G}_2 &= \frac{c(\gamma+1)}{(\alpha-1)(\alpha-2)(\delta-1)(\delta-2)}\mathcal{G}_2(\alpha-2,\beta,\gamma+2,\delta-2;x,y)\\ &+\frac{2}{(\alpha-1)(\beta-1)}\mathcal{G}_2(\alpha-1,\beta-1,\gamma,\delta;x,y)\\ &+\frac{\delta(\delta+1)}{(\beta-1)(\beta-2)(\gamma-1)(\gamma-2)}\mathcal{G}_2(\alpha,\beta-2,\gamma-2,\delta+2;x,y),\quad \alpha,\beta,\gamma,\delta\neq 1,2\end{aligned} \tag{4.14}$$

and

$$\begin{aligned}\breve{\mathbf{I}}^2\mathcal{G}_3 &= \frac{a(\alpha+1)}{(\beta-1)(\beta-2)(\beta-3)(\beta-4)}\mathcal{G}_3(\alpha+2,\beta-4;x,y)\\ &+\frac{2}{(\alpha-1)(\beta-1)}\mathcal{G}_3(\alpha-1,\beta-1;x,y)\\ &+\frac{\beta(\beta+1)}{(\alpha-1)(\alpha-2)(\alpha-3)(\alpha-4)}\mathcal{G}_3(\alpha-4,\beta+2;x,y),\quad \alpha,\beta\neq 1,2,3,4.\end{aligned} \tag{4.15}$$

Proof Acting by this integral operator $\breve{\mathbf{I}}$ on the Horn's function $\mathcal{G}_1$, it follows that

$$\begin{aligned}\breve{\mathbf{I}}\mathcal{G}_1 &= \sum_{m=1,n=0}^{\infty}\frac{(\alpha)_{m+n}(\beta)_{n-m}(\gamma)_{m-n}}{(m+1)!n!}x^{m+1}y^n\\ &\quad+\sum_{m=0,n=1}^{\infty}\frac{(\alpha)_{m+n}(\beta)_{n-m}(\gamma)_{m-n}}{m!(n+1)!}x^m y^{n+1}\\ &=\frac{\beta}{(\alpha-1)(\gamma-1)}\mathcal{G}_1(\alpha-1,\beta,\gamma-1;x,y)\\ &\quad+\frac{\gamma}{(\alpha-1)(\beta-1)}\mathcal{G}_1(\alpha-1,\beta-1,\gamma+1;x,y),\alpha,\beta,\gamma\neq 1.\end{aligned}$$

By repeating the integral operator again for Horn function $\mathcal{G}_1$, we get (4.13). In similar way, we obtain the Eqs. (4.14) and (4.15)

Theorem 4.5 *The connections between integral and differential operators of $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ are*

$$\breve{\mathbf{I}}^{\ell}\mathcal{G}_1 = \frac{1}{(1-\alpha)_{2\ell}}\prod_{k=1}^{\ell}\big(\Theta_x+\Theta_y-k+1\big)\mathcal{G}_1(\alpha-2\ell,\beta,\gamma;x,y),\quad \alpha\neq 1,2,3,\ldots, \tag{4.16}$$

$$\breve{\mathbf{I}}^{\ell}\mathcal{G}_2 = \frac{1}{(1-\alpha)_{2\ell}}\prod_{k=1}^{\ell}\big(\Theta_x+\Theta_y-k+1\big)\mathcal{G}_2(\alpha-2\ell,\beta,\gamma,\delta;x,y), \quad \alpha\neq 1,2,3,\ldots \tag{4.17}$$

and

$$\breve{\mathbf{I}}^{\ell}\mathcal{G}_3 = \frac{1}{(1-\alpha)_{\ell}(1-\beta)_{\ell}}\prod_{k=1}^{\ell}\big(\Theta_x+\Theta_y-k+1\big)\mathcal{G}_3 (\alpha-\ell,\beta-\ell;x,y), \quad \alpha,\beta\neq 1,2,3,\ldots. \tag{4.18}$$

Proof By using the integral operator and differential operators for $\mathcal{G}_1$, we get

$$\breve{\mathbf{I}}\mathcal{G}_1 = \frac{1}{(\alpha-1)(\alpha-2)}\big(\Theta_x+\Theta_y\big)\mathcal{G}_1(\alpha-2,\beta,\gamma;x,y), \quad \alpha\neq 1,2.$$

Iteration the above relation for ℓ times, implies (4.16). In similar way, we obtain the Eqs. (4.17) and (4.18)

Theorem 4.6 *We have the integral operators $\breve{\mathbf{I}}_x^{\ell}$ and $\breve{\mathbf{I}}_y^{\ell}$ for Horn functions $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$:*

$$\breve{\mathbf{I}}_x^{\ell}\mathcal{G}_1 = \frac{(\beta)_{\ell}}{(1-\alpha)_{\ell}(1-\gamma)_{\ell}}\mathcal{G}_1(\alpha-\ell,\beta+\ell,\gamma-\ell;x,y), \quad \alpha,\gamma\neq 1,2,3,\ldots, \tag{4.19}$$

$$\breve{\mathbf{I}}_y^{\ell}\mathcal{G}_1 = \frac{(\gamma)_{\ell}}{(1-\alpha)_{\ell}(1-\alpha)_{\ell}}\mathcal{G}_1(\alpha-\ell,\beta-\ell,\gamma+\ell;x,y), \quad \alpha,\beta\neq 1,2,3,\ldots, \tag{4.20}$$

$$\begin{aligned}\breve{\mathbf{I}}_x^{\ell}\mathcal{G}_2 &= \frac{(\gamma)_{\ell}}{(1-\alpha)_{\ell}(1-\delta)_{\ell}}\mathcal{G}_2(\alpha-\ell,\beta,\gamma+\ell,\delta-\ell;x,y), \quad \alpha,\delta\neq 1,2,3,\ldots,\\ \breve{\mathbf{I}}_y^{\ell}\mathcal{G}_2 &= \frac{(\delta)_{\ell}}{(1-\beta)_{\ell}(1-\gamma)_{\ell}}\mathcal{G}_2(\alpha,\beta-\ell,\gamma-\ell,\delta+\ell;x,y), \quad \alpha,\beta,\gamma\neq 1,2,3,\ldots\end{aligned} \tag{4.21}$$

and

$$\begin{aligned}\breve{\mathbf{I}}^{\ell}\ \mathcal{G}_3 &= \frac{(-1)^{\ell}(\alpha)_{\ell}}{(1-\beta)_{2\ell}}\mathcal{G}_3(\alpha+\ell,\beta-2\ell;x,y), \quad \beta\neq 1,2,3,\ldots,\\ \breve{\mathbf{I}}^{\ell}\ \mathcal{G}_3 &= \frac{(-1)^{\ell}(\beta)_{\ell}}{(1-\alpha)_{2\ell}}\mathcal{G}_3(\alpha-2\ell,\beta+\ell;x,y), \quad \alpha\neq 1,2,3,\ldots.\end{aligned} \tag{4.22}$$

Proof Now, we consider the integral operator $\breve{\mathbf{I}}_x^{\ell}$, where $\breve{\mathbf{I}}_x = \int_0^x dx$ such that $\breve{\mathbf{I}}_x^{\ell} = \breve{\mathbf{I}}_x I_x^{\widehat{\ell-1}}$, we get

$$\breve{\mathbf{I}}_x\mathcal{G}_1 = \frac{\beta}{(\alpha-1)(\gamma-1)}\mathcal{G}_1(\alpha-1,\beta+1,\gamma-1;x,y), \quad \alpha,\gamma\neq 1.$$

Iteration the above formula for ℓ times, implies (4.19). Similarly the integral operator $\breve{\mathbf{I}}_y^{\ell}$, the relations (4.20)–(4.22) can be proved. □

Theorem 4.7 *The following integral for Horn's hypergeometric functions $\mathcal{G}_2$ holds true:*

$$\mathcal{G}_2 = \frac{1}{\Gamma(\alpha)\Gamma(\beta)}\int_0^{\infty}\int_0^{\infty} e^{-t-u}t^{\alpha-1}u^{\beta-1}e^{\frac{u^2x}{t}}e^{\frac{t^2y}{u}}\,dtdu. \tag{4.23}$$

Proof Starting from (1.9) and using the gamma function, we obtain the integral for $\mathcal{G}_2$ (4.23). □

5 Infinite summation for $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$

Here, we derive some infinite summation for $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$.

Theorem 5.1 *For $|\mathbf{t}| < 1$, the infinite summation for Horn's functions $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$*

$$\sum_{\ell=0}^{\infty} \frac{(\alpha)_\ell}{\ell!} \mathcal{G}_1(\alpha+\ell, \beta, \gamma; x, y)\mathbf{t}^\ell = (1-\mathbf{t})^{-\alpha} \mathcal{G}_1\left(\alpha, \beta, \gamma; \frac{x}{1-\mathbf{t}}, \frac{y}{1-\mathbf{t}}\right), \tag{5.1}$$

$$\sum_{\ell=0}^{\infty} \frac{(\beta)_\ell}{\ell!} \mathcal{G}_1(\alpha, \beta+\ell, \gamma; x, y)\mathbf{t}^\ell = (1-\mathbf{t})^{-\beta} \mathcal{G}_1\left(\alpha, \beta, \gamma; x(1-\mathbf{t}), \frac{y}{1-\mathbf{t}}\right),$$

$$\sum_{\ell=0}^{\infty} \frac{(\gamma)_\ell}{\ell!} \mathcal{G}_1(\alpha, \beta, \gamma+\ell; x, y)\mathbf{t}^\ell = (1-\mathbf{t})^{-\gamma} \mathcal{G}_1\left(\alpha, \beta, \gamma; \frac{x}{1-\mathbf{t}}, y(1-\mathbf{t})\right), \tag{5.2}$$

$$\sum_{\ell=0}^{\infty} \frac{(\alpha)_\ell}{\ell!} \mathcal{G}_2(\alpha+\ell, \beta, \gamma, \delta; x, y)\mathbf{t}^k = (1-\mathbf{t})^{-\alpha} \mathcal{G}_2\left(\alpha, \beta, \gamma, \delta; \frac{x}{1-\mathbf{t}}, y\right),$$

$$\sum_{\ell=0}^{\infty} \frac{(\beta)_\ell}{\ell!} \mathcal{G}_2(\alpha, \beta+\ell, \gamma, \delta; x, y)\mathbf{t}^\ell = (1-\mathbf{t})^{-\beta} \mathcal{G}_2\left(\alpha, \beta, \gamma, \delta; x, \frac{y}{1-\mathbf{t}}\right),$$

$$\sum_{\ell=0}^{\infty} \frac{(\gamma)_\ell}{\ell!} \mathcal{G}_2(\alpha, \beta, \gamma+\ell, \delta; x, y)\mathbf{t}^\ell = (1-\mathbf{t})^{-\gamma} \mathcal{G}_2\left(\alpha, \beta, \gamma, \delta; x(1-\mathbf{t}), \frac{y}{1-\mathbf{t}}\right),$$

$$\sum_{\ell=0}^{\infty} \frac{(\delta)_\ell}{\ell!} \mathcal{G}_2(\alpha, \beta, \gamma, \delta+\ell; x, y)\mathbf{t}^\ell = (1-\mathbf{t})^{-\delta} \mathcal{G}_2\left(\alpha, \beta, \gamma, \delta; \frac{x}{1-\mathbf{t}}, y(1-\mathbf{t})\right) \tag{5.3}$$

and

$$\sum_{\ell=0}^{\infty} \frac{(\alpha)_\ell}{\ell!} \mathcal{G}_3(\alpha+\ell, \beta; x, y)\mathbf{t}^\ell = (1-\mathbf{t})^{-\alpha} \mathcal{G}_3\left(\alpha, \beta; x(1-\mathbf{t}), \frac{y}{(1-\mathbf{t})^2}\right),$$

$$\sum_{\ell=0}^{\infty} \frac{(\beta)_\ell}{\ell!} \mathcal{G}_3(\alpha, \beta+\ell; x, y)\mathbf{t}^\ell = (1-\mathbf{t})^{-\beta} \mathcal{G}_3\left(\alpha, \beta; \frac{x}{(1-\mathbf{t})^2}, y(1-\mathbf{t})\right) \tag{5.4}$$

are established.

Proof With the help of the fact that

$$(1-\mathbf{t})^{-\alpha} = \sum_{\ell=0}^{\infty} \frac{(\alpha)_\ell}{\ell!} \mathbf{t}^\ell, \ |\mathbf{t}| < 1,$$

expanding the sums, collecting the expression of the right-hand side (5.1) and after some simplification, we obtain (5.1). In a similar manner, the infinite summation formulas (5.2)–(5.4) can be established.

6 Concluding remarks

As a brief consequence of the numerous new recursion formulas, integral operators, infinite summation formulas and interesting results of Horn functions $\mathcal{G}_1$, $\mathcal{G}_2$ and $\mathcal{G}_3$ which we have discussed, an analytic procedure to calculate many of the various outcomes has been established. Our analytic expressions can be used as a benchmark for the accuracy of unique approximation approaches designed mainly for the investigation of radiation discipline problems.

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