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Some generalized inequalities involving extended beta and gamma functions for several variables

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Abstract

Recently, extensions of the gamma and beta functions have been studied due to their appealing properties and wide range of applications in various scientific fields. This note aims to investigate generalized inequalities associated with the extended beta and gamma functions.

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1 Introduction

In various branches of pure and applied mathematics, special functions have become essential tools for scientists and engineers. Among these, the gamma and beta functions are particularly notable. The gamma function, first introduced by Swiss mathematician Leonhard Euler, appears in numerous mathematical contexts, including Riemann's zeta function, asymptotic series, definite integrals, hypergeometric series, and number theory. Due to its importance, the gamma function has been studied by several renowned mathematicians, such as Adrien-Marie Legendre (1752–1833), Carl Friedrich Gauss (1777–1855), Christoph Gudermann (1798–1852), and Joseph Liouville (1809–1882). The gamma function is classified as a special transcendental function.

The gamma and beta functions are fundamental in various fields of mathematics and applied sciences, serving as essential tools in probability theory, statistics, and mathematical modeling. Their applications are extensive, ranging from statistical distributions to solutions of differential equations.

The gamma function generalizes the factorial function and is crucial in defining various probability distributions, including the gamma and beta distributions, which are widely used in statistics for modeling continuous random variables. For instance, the noncentral gamma distribution plays a significant role in radar detection and communications, highlighting the gamma function's relevance in practical applications (Segura [1]). Additionally, the beta function is pivotal in Bayesian statistics and is often used to model random variables that are constrained to an interval (Segura [1]).

In mathematical modeling, the gamma and beta functions are used to solve fractional differential equations, which are essential in physics and engineering. Recent studies have introduced generalized forms of these functions, further extending their applicability in fractional calculus (Ata and Kıymaz [2], Matouk [3]). For example, the generalized gamma function has been applied to model biological systems, demonstrating its interdisciplinary significance (Matouk [3]).

The applications of the gamma and beta functions extend to matrix theory, where matrix versions of these functions have been developed. These extensions facilitate the study of special matrix functions and their applications in statistics and differential equations (Zou *et al.* [4], He *et al.* [5]). The introduction of k-gamma and k-beta functions has further broadened the scope of their applications, particularly in high-energy physics and mathematical analysis (Khammash *et al.* [6], Tassaddiq [7]).

Moreover, the gamma function's role in asymptotic analysis and integral representations is noteworthy. It has been used to derive series representations for the incomplete gamma function, which are crucial for evaluating integrals in various mathematical contexts (Amore [8]). The analytical properties of the gamma function also contribute to the evaluation of integrals involving Laplace and Fourier transforms, underscoring its importance in theoretical and applied mathematics (Iddrisu and Tetteh [9], Choi and Srivastava [10]).

Due to the variety of applications of beta and gamma functions, many researchers have derived their representations and properties. Diaz *et al.* [11–13] provided integral representations of k-beta and k-gamma functions and derived their properties. They also provided a representation for the Pochhammer k-symbol. After Diaz *et al.* [11–13], many other researchers, including Kokologiannaki [14, 15], Krasiniqi [16], Mansour [17] and Mubeen *et al.* [18], added their contributions, making these functions more interesting and useful. Mubeen *et al.* [19] discussed a representation of the k-beta and k-gamma functions. Golub [20] also contributed to this framework. Mubeen and Habibullah [21] provided integral representations of some generalized confluent hypergeometric k-function using properties of the Pochhammer k-symbol, k-beta and k-gamma functions. Mubeen *et al.* [22] studied other extensions of the k-beta and k-gamma functions involving a confluent hypergeometric k-function. Mubeen [23] introduced a k-analogue of Kommer's formula and evaluated some useful results using hypergeometric k-functions. Rehman *et al.* [24] introduced a beta k-function for several variables. They also extended the beta k-function for n variables. Mubeen and Habibullah [25] defined the k fractional integration and gave its application.

Inequalities involving extended gamma or beta functions have not been studied much. The papers known to us are as follows. Rehman *et al.* [26, 27] derived several inequalities involving k-beta and k-gamma functions. Raissouli and Soubhy [28] studied some inequalities involving two generalized beta functions in n variables. This note presents the most general inequalities involving extended beta and gamma functions. By expanding on previous research and offering fresh perspectives, we uncover numerous new insights and advancements in the field. Our study covers various inequality categories, including integral inequalities, functional inequalities, and those designed for specific parameters or functions. Through a comprehensive comparison of our results with established findings, we effectively emphasize the originality and importance of our contributions. Our goal is to enrich the existing knowledge in this area and further the understanding and practical applications of extended beta and gamma functions.

The k-beta and k-gamma functions have the following standard representations [19]

$$\beta_k(\phi,\psi) = \frac{1}{k}\int_0^1 m^{\frac{\phi}{k}-1}(1-m)^{\frac{\psi}{k}-1}dm, \qquad where \qquad Re(\phi)>0, \qquad Re(\psi)>0$$

and

$$\Gamma_k(\phi) = \int_0^\infty m^{\phi-1}e^{-\frac{m^k}{k}}dm, \qquad \text{where} \quad Re(\phi)>0, \qquad k>0.$$

Some other extensions of extended k-beta and k-gamma functions described in [14–18] are

$$\beta_k(\phi,\psi;a) = \frac{1}{k}\int_0^1 m^{\frac{\phi}{k}-1}(1-m)^{\frac{\psi}{k}-1}e^{-\frac{a^k}{km(1-m)}}dm,$$

$$\beta_k(\phi,\psi;a,b) = \frac{1}{k}\int_0^1 m^{\frac{\phi}{k}-1}e^{\frac{-a^k}{m}}(1-m)^{\frac{\psi}{k}-1}e^{-\frac{b^k}{k(1-m)}}dm \tag{1}$$

and

$$\Gamma_{a,k}(\phi) = \int_0^\infty m^{\phi-1}e^{-\frac{m^k}{k}}e^{-\frac{a^k}{km^k}}dm, \tag{2}$$

where $Re(\phi)>0$ and $Re(\psi)>0$. The generalized beta and gamma functions can be expressed as [22]

$$\beta_{a,k}^{(a_n,b_n)}(\phi,\psi) = \frac{1}{k}\int_0^1 m^{\frac{\phi}{k}-1}(1-m)^{\frac{\psi}{k}-1}\ {}_1F_{1,k}\left(a_n,b_n;-\frac{a^k}{km(1-m)}\right)dm \tag{3}$$

and

$$\Gamma_k^{(a_n,b_n)}(\phi,a) = \int_o^\infty m^{\phi-1}\ {}_1F_{1,k}\left(a_n;b_n;-\frac{m^k}{k}-\frac{a^k}{km^k}\right)dm, \qquad where \quad a,b,>0. \tag{4}$$

where ${}_1F_{1,k}$ is the confluent hypergeometric function [21] defined by

$${}_1F_{1,k}\left(a_n,b_n;l\right) = \sum_{m=0}^{\infty}\frac{(a_n)_{m,k}}{(b_n)_{m,k}}\cdot\frac{l^m}{m!}. \tag{5}$$

If $a_n>0$, $b_n-a_n>0$ and $k>0$, then we have the following integral representation

$${}_1F_{1,k}\left(a_n,b_n;l\right) = \frac{1}{k}\frac{\Gamma\left(b_n\right)}{\Gamma\left(a_n\right)\Gamma\left(b_n-a_n\right)}\int_0^1 u^{a_n-1}(1-u)^{b_n-a_n-1}e^{lu}du. \tag{6}$$

By making the substitution, we obtain

$$ {}_1F_{1,k}\left(a_n; b_n; l\right) = e^l\ {}_1F_{1,k}\left(b_n - a_n; b; -l\right). \tag{7}$$

Remark 1.1 The real map is strictly increasing and strictly convex on $\Re$. It follows that ${}_1F_{1,k}\left(a_n; b_n; l\right) \geq {}_1F_{1,k}\left(a_n; b_n; 0\right)$ for any $l \geq 0$ and $0 \leq {}_1F_{1,k}\left(a_n; b_n; l\right) \leq 1$ for any $l \leq 0$.

In [24], the authors introduced the k-beta function for more then two variables and provided some useful representations. Let $n > 3$ be an integer and $E_{(n-1)}$ be the $(n-1)$ simplex of $\Re^{n-1}$ described as

$$E_{(n-1)} = \left[\left(m_1, \ldots, m_{(n-1)}\right) \in \Re^{(n-1)} : \sum_{i=1}^{n-1} m_i \leq 1; m_i \geq 0;\ \text{for } i = 1, \ldots, n-1\right].$$

The beta function involving n variables $\phi_1, \ldots, \phi_n > 0$ is

$$\beta_k\left(\phi_1, \ldots, \phi_n\right) = \frac{1}{k^{n-1}} \int_{(E_{n-1})} \prod_{i=1}^{n} m_i^{\frac{\phi_i}{k}}\, dm_1 \cdots dm_{n-1}. \tag{8}$$

Let

$$m_n = 1 - \sum_{i=1}^{n-1} m_i$$

and

$$\sigma(l) = \sum_{i=1}^{n} (\phi_i),$$

then (8) can be written as

$$\beta_k\left(\phi_1, \ldots, \phi_n\right) = \frac{\prod_{i=1}^{n} \Gamma_k\left(\phi_i\right)}{\Gamma_k\left(\sigma(\phi)\right)}.$$

Some other representations of the extended k-beta function are

$$\beta_k\left(\phi_1, \ldots, \phi_n; a\right) = \frac{1}{k^{n-1}} \int\limits_{(E_{N-1})} \prod_{i=1}^{n} m_i^{\frac{\phi_i}{k}-1} e^{\frac{-a^k}{k\pi(m)}}\, dm_1 \cdots dm_{n-1} \tag{9}$$

and

$$\beta_k\left(\phi_1, \ldots, \phi_n; a_1, \ldots, a_n\right) = \frac{1}{k^{n-1}} \int\limits_{(E_{N-1})} \prod_{i=1}^{n} m_i^{\frac{\phi_i}{k}-1} e^{\frac{-a_i^k}{k\pi(m)}}\, dm_1 \cdots dm_{n-1} \tag{10}$$

for any $\phi_1, \ldots, \phi_n > 0$.

Now, we defined the k-gamma function for the several variables. Let $\phi =: (\phi_1, \dots, \phi_n) > 0$, $\alpha =: (\alpha_1, \dots, \alpha_n) > 0$, $\beta =: (\beta_1, \dots, \beta_n)$ and $c = (c_1, \dots, c_n) \geq 0$. The generalized k-gamma function is defined by

$$\Gamma_{k,c}(\phi) = \int\limits_{(0,\infty)^n} \prod_{i=1}^{n} m_i^{\phi_i - 1} e^{-\frac{m_i^k}{k}} e^{-\frac{c_i^k}{k m_i^k}} \, dm. \tag{11}$$

If $c = 0$, then

$$\Gamma_{k,0}(\phi) = \prod_{i=1}^{n} \Gamma_k(\phi_i).$$

Another representation of (11) is

$$\Gamma_{k,c}^{(\alpha_n,\beta_n)}(\phi) = \int\limits_{(0,\infty)^n} \prod_{i=1}^{n} m_i^{\phi_i - 1} \, {}_1F_{1,k}\left((\alpha_n)_i, (\beta_n)_i, -\frac{m_i^k}{k} - \frac{c_i^k}{k m_i^k} \right) dm. \tag{12}$$

If $c = 0$, then

$$\Gamma_{k,0}^{(\alpha_n,\beta_n)}(l) = \int\limits_{(0,\infty)^n} \prod_{i=1}^{n} m_i^{\phi_i - 1} \, {}_1F_1\left((\alpha_n)_i ; (\beta_i)_n ; -\frac{m_i^k}{k} \right) dm. \tag{13}$$

If $n = 1$, then (11) and (12) reduce to (2) and (4), respectively. If $\alpha_n = \beta_n$, then (12) becomes (11).

2 Main results

In this section, we study the generalized k-beta function of the first kind.

2.1 Generalized beta k function of the first kind

Definition 2.1 Let $\phi = (\phi_1, \dots, \phi_n) > 0$, $a_n > 0$, $b_n > 0$, $\eta > 0$ and $\zeta \geq 0$. The generalized k-beta function of first kind is

$$\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\zeta) =: \frac{1}{k^{n-1}} \int\limits_{E_{n-1}} \prod_{i=1}^{n} t_i^{\frac{\phi_i}{k} - 1} \, {}_1F_{1,k}\left(a_n, b_n; -\frac{\eta^k}{k\pi(t)} - \zeta^k \frac{\pi(t)}{\eta^k} \right) dt, \tag{14}$$

where $dt =: dt_1 \cdots dt_{n-1}$ and $\pi(t) =: \prod_{i=1}^{n} t_i$ with $t_n = 1 - \sum_{i=1}^{n-1} t_i$. If $\zeta = 0$, we obtain

$$\beta_{0,k}^{(a_n,b_n)}(\phi,\eta) = \frac{1}{k^{n-1}} \int\limits_{(E_n - 1)} \prod_{i=1}^{n} t_i^{\frac{\phi_i}{k} - 1} \, {}_1F_{1,k}\left(a_n, b_n; -\frac{\eta^k}{k\pi(t)} \right) dt. \tag{15}$$

If $n = 2$ and $\zeta = 0$, then (14) becomes (3). If $a_n = b_n$, then (14) is (9).

Proposition 2.1 *Let $\phi = (\phi_1, \dots, \phi_n) > 0$, $a_n > 0$, $b_n > 0$, $\eta > 0$ and $\zeta \geq 0$ with $b_n - a_n > 0$. Then, $0 \leq \beta_{\zeta,k}^{(a_n,b_n)} \leq \beta_k(\phi)$ and so $\beta_{\eta,k}^{(a_n,b_n)}(\phi;\eta)$ is well-defined.*

Proof With the help of Remark 1.1, we have

$$0 \le \prod_{i=n}^{n} t_i^{\frac{\phi_i}{k}-1} \; {}_1F_{1,k}\left(a_n, b_n; -\frac{\eta^k}{k\pi(t)} - \zeta^k \frac{k\pi(t)}{\eta^k}\right) \le \prod_{i=1}^{n} t_i^{\frac{\phi_i}{k}-1}. \tag{16}$$

Integrating (16) over $t \in E_{n-1}$ and using (8) and (14), we obtain the required result. □

Now, we discuss our main result, which consists of several generalized inequalities involving $\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\zeta)$.

Theorem 2.1 *Let $a_n > 0$, $b_n - a_n > 0$, $\eta > 0$ and $\zeta \ge 0$. Then*

$$\left(\beta_{\zeta,k}^{(a_n,b_n)}(\phi+\psi;\eta)\right)^2 \le \beta_{\zeta,k}^{(a_n,b_n)}(2\phi;\eta)\beta_{\zeta,k}^{(a_n,b_n)}(2\psi;\eta) \tag{17}$$

holds for any $\phi, \psi \in (0,\infty)^n$. The real valued function $\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta)$ is convex on $(0,\infty)^n$.

Proof Let $\phi = (\phi_1, \dots, \phi_n)$, $\psi = (\psi_1, \dots, \psi_n)$ and

$$\lambda(t) = {}_1F_{1,k}\left(a_n, b_n, -\frac{\eta^k}{k\pi(t)} - \zeta^k \frac{\pi(t)}{\eta^k}\right).$$

As $\lambda(t) \ge 0$, we can write

$$\begin{aligned}&\left(\beta_{\zeta,k}^{(a_n,b_n)}(\phi+\psi;\eta)\right)^2 \\ &= \frac{1}{k^{n-1}}\left(\int_{E_{n-1}} \left(\prod_{i=1}^{n} t_i^{\frac{\phi_i}{k}-\frac{1}{2}}\right)(\lambda(t))^{\frac{1}{2}}\left(\prod_{i=1}^{n} t_i^{\frac{\psi_i}{k}-\frac{1}{2}}(\lambda(t))^{\frac{1}{2}}\right)dt\right)^2.\end{aligned}$$

By the Cauchy-Schwartz inequalities, (17) is obtained.

Now, we have an interesting remark that (17) is equivalent to

$$\beta_{\zeta,k}^{(a_n,b_n)}\left(\frac{\phi+\psi}{2};\eta\right) \le \left(\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta)\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta)\right)^{\frac{1}{2}}.$$

Using arithmetic-geometric mean inequality $\sqrt{\phi\psi} \le \frac{1}{2}\phi + \frac{1}{2}\psi$,

$$\beta_{\zeta,k}^{(a_n,b_n)}\left(\frac{\phi+\psi}{2};\eta\right) \le \frac{1}{2}\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta) + \frac{1}{2}\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta).$$

This expression shows that $\phi \longmapsto \beta_{\zeta,k}^{(a_n,b_n)}$ is mid convex. In addition to the fact that $\phi \longmapsto \beta_{\zeta,k}^{(a_n,b_n)}$ has the continuous property, these facts ensure $\phi \longmapsto \beta_{\zeta,k}^{(a_n,b_n)}$ is convex. The proof is complete. □

Lemma 2.1 *Let $\phi > 0$, $\psi > 0$. Then, we have a real valued function $w \longmapsto {}_1F_{1,k}(a_n; b_n; w)$, which is differentiable on $\Re$ and*

$$\frac{d}{dw}\,{}_1F_{1,k}(a_n; b_n; w) = \frac{a_n}{b_n}\,{}_1F_{1,k}(a_n + k; b_n + k; w). \tag{18}$$

If $w = 0$, we have

$$\frac{d}{dw}\,{}_1F_{1,k}\,(a_n; b_n; 0) = \frac{a_n}{b_n}. \tag{19}$$

Now, we state the following result.

Theorem 2.2 *Let $a_n > 0$, $b_n - a_n > 0$, $\eta > 0$ and $\zeta \geq 0$. Then,*

$$\begin{aligned} &\beta_k^{(a_n,b_n)}(\phi;\eta) - \frac{a_n}{b_n}\frac{\zeta^k}{\eta^k}\beta_k^{(a_n+1,b_n+1)}(\phi + ke;\eta) \\ &\leq \beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta) \leq \beta_k^{(a_n,b_n)}(\phi;\eta) \leq \beta_k(\phi) \end{aligned}$$

is valid for all $\phi \in (0,\infty)^n$, and $e =: (1,1,\dots,1)$.

Proof Using Remark 1.1, (14) and (15), we have $\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta) \leq \beta_k^{(a_n,b_n)}(\phi;\eta) \leq \beta_k(\phi)$. Let prove the next part of the inequality

$$\beta_k^{(a_n,b_n)}(\phi;\eta) - \frac{a_n}{b_n}\frac{\zeta^k}{\eta^k}\beta_k^{(a_n+1,b_n+1)}(\phi + ke;\eta) \leq \beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta) \tag{20}$$

for fixed $a_n, b_n - a_n > 0$, and the map $b \longmapsto {}_1F_{1,k}\,(a_n; b_n; b)$ is convex on $\Re$. We also have $f : \Re \longrightarrow \Re$ a convex function, differentiable at c_0, so $f(c) \geq f\,(c_0) + (c - c_0)\acute{f}\,(c_0)$. Applying this and (18), we have

$$\begin{aligned} &{}_1F_{1,k}\left(a_n; b_n; -\frac{\eta^k}{k\pi(t)} - \zeta^k\frac{\pi(t)}{\eta^k}\right) \\ &\geq {}_1F_{1,k}\left(a_n; b_n; -\frac{\eta^k}{k\pi(t)}\right) - \frac{a_n}{b_n}\frac{\zeta^k}{\eta^k}\pi(t)\,{}_1F_{1,k}\left(a_n + 1; b_n + 1; -\frac{\eta^k}{k\pi(t)}\right). \end{aligned} \tag{21}$$

Next, multiplying (21) by $\prod_{i=1}^n t^{\frac{\phi_i}{k}-1}$, integrating over $t \in E_{n-1}$, and using (14) and (15), we have (20). □

Proposition 2.2 *Let $a_n > 0$, $b_n - a_n > 0$, $\eta > 0$ and $\zeta \geq 0$. For $\phi \in (1,\infty)^n$, we have*

$$\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta) \geq \beta(\phi,k) - \frac{a_n}{b_n}\frac{\eta^k}{k}\beta_k(\phi - ke) - \frac{a_n}{b_n}\frac{\zeta^k}{\eta^k}\beta_k(\phi + ke)$$

with $e = (1,1,\dots,1)$.

Proof Using (19), we have

$${}_1F_{1,k}\left(a_n; b_n; -\frac{\eta^k}{k\pi(t)} - \zeta^k\frac{\pi(t)}{\eta^k}\right) \geq {}_1F_{1,k}(a_n; b_n; 0) + \frac{a_n}{b_n}\left(-\frac{\eta^k}{k\pi(t)} - \zeta^k\frac{\pi(t)}{\eta^k}\right). \tag{22}$$

Multiplying (22) by $\prod_{i=1}^n t^{\frac{\phi_i}{k}-1}$, the rest of the proof is similar to that of Theorem 2.2. □

Lemma 2.2 *We have*

$$\sup_{t\in E_{n-1}} \pi(t) = x^{-x}.$$

Proof Take n positive real numbers $c_1, c_2, \dots, c_n$ and use the arithmetic-geometric mean inequality $\sqrt[n]{c_1 \cdots c_n} \le \frac{c_1 + \dots + c_n}{n}$. Letting $c_1 = t_1, c_2 = t_2, \dots, c_{n-1} = t_{n-1}, c_n = t_n =: 1 - t_1 - \dots - t_{n-1}$ in the inequality, we have $\sqrt[n]{\pi(t)} \le \frac{1}{n}$ or $\pi(t) \le n^{-n}$ for any $t \in E_{n-1}$. This inequality is an equality for $(t_1, \dots, t_{n-1}) = \left(\frac{1}{n}, \dots, \frac{1}{n}\right)$. □

We now prove a refinement of the inequality $\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta) \le \beta_k(\phi)$.

Theorem 2.3 *Let $a_n > 0$, $b_n - a_n > 0$, $\eta > 0$, $\zeta \ge 0$ and $\phi \in (0,\infty)^n$. Also assume that $\eta \ge \sqrt{\zeta}$. Then,*

$$\frac{\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta)}{\beta_k(\phi)} \le {}_1F_{1,k}\left(a_n; b_n; -\frac{\eta^k}{k} n^n - \frac{\zeta^k}{\eta^k n^n}\right) \le {}_1F_{1,k}\left(a_n; b_n; -2\sqrt{\zeta}\right) \le 1, \tag{23}$$

where $\beta_k(\phi)$ is as defined in (8).

Proof First, we consider the inequality

$${}_1F_{1,k}\left(a_n; b_n; -\frac{\eta^k}{k} n^n - \frac{\zeta^k}{\eta^k n^n}\right) \le {}_1F_{1,k}\left(a_n; b_n; -2\sqrt{\zeta}\right) \le 1.$$

We are able to verify it easily by considering

$$-\frac{\eta^k}{k} n^n - \frac{\zeta^k}{\eta^k n^n} \le -2\sqrt{\zeta} \le 0, \qquad {}_1F_{1,k}\left(a_n; b_n; 0\right) = 1$$

and $w \longmapsto {}_1F_{1,k}\left(a_n; b_n; w\right)$ is an increasing real valued function. We will prove the next part of the inequality

$$\frac{\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta)}{\beta_k(\phi)} \le {}_1F_{1,k}\left(a_n; b_n; -\frac{\eta^k}{k} n^n - \frac{\zeta^k}{\eta^k n^n}\right). \tag{24}$$

Using (14), we obtain

$$\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta) \le \beta_k(\phi) \sup_{E_{n-1}} {}_1F_{1,k}\left(a_n; b_n; -\frac{\eta^k}{k\pi(t)} - \zeta^k \frac{\pi(t)}{\eta^k}\right). \tag{25}$$

The supremum exists, and

$$\begin{aligned} {}_1F_{1,k}\left(a_n; b_n; -\frac{\eta^k}{k\pi(t)} - \zeta^k \frac{\pi(t)}{\eta^k}\right) &= \frac{\Gamma(b_n)}{k\Gamma(a_n)\Gamma(b_n - a_n)} \\ &\quad \cdot \int_0^1 b^{\frac{a_n}{k}-1}(1-b)^{\frac{b_n-a_n}{k}-1} \exp\left[b\left(-\frac{\eta^k}{k\pi(t)} - \zeta^k \frac{\pi(t)}{\eta^k}\right)\right] db. \end{aligned} \tag{26}$$

It is sufficient to find an upper bond of $t \mapsto -\frac{\eta^k}{k\pi(t)} - \zeta^k \frac{\pi(t)}{\eta^k}$. Since $b \in [0,1]$ and the exponential function is increasing, if $\eta \ge \sqrt{\zeta}$ and $0 < u < 1$, then

$$\sup_{0 \le s \le u} \left(-\frac{\eta^k}{ks} - \zeta^k \frac{s}{\eta^k}\right) = -\frac{\eta^k}{ku} - \zeta^k \frac{u}{\eta^k}. \tag{27}$$

By Lemma 2.2, $\pi(t) \leq x^{-x}$ and so $E_{n-1} \subset \{(t_1, \ldots, t_{n-1}) : 0 \leq \pi(t) \leq x^{-x}\} =: G$. This with (27) implies that

$$\begin{aligned}\sup_{t\epsilon E_{n-1}} \left(-\frac{\eta^k}{k\pi(t)} - \zeta^k \frac{\pi(t)}{\eta^k}\right) &\leq \sup_{t\in G} \left(-\frac{\eta^k}{k\pi(t)} - \zeta^k \frac{\pi(t)}{\eta^k}\right) \\ &= \sup_{0\leq s\leq x^{-x}} \left(-\frac{\eta^k}{ks} - \zeta^k \frac{s}{\eta^k}\right) \\ &= \frac{-\eta^k}{k} n^n - \frac{\zeta^k}{\eta^k n^n}.\end{aligned}$$

Substituting into (26) and using (25), we have (24). □

Corollary 2.1 *Under the assumptions of Theorem 2.3 for any $z > 0$, we have*

$$\int_0^\infty \eta^{z-1} \beta_k^{(a_n,b_n)}(\phi;\eta) d\eta \leq \beta_k(\phi) n^{(-n^{-n})^{\frac{z}{k}}} \Gamma_k^{(a_n,b_n)}(z), \tag{28}$$

where $\Gamma_k^{(a_n,b_n)}(z)$ is defined in (13). If $z = 1$

$$\int_0^\infty \beta_k^{(a_n,b_n)}(\phi;\eta) d\eta \leq \beta_k(l) n^{-n^{\frac{1}{k}}} \Gamma^{(a_n,b_n)}(1).$$

Proof Taking $\zeta = 0$ in (23), we have

$$\beta_k^{(a_n,b_n)}(\phi;\eta) \leq \beta_k(\phi)\, {}_1F_{1,k}\left(a_n; b_n; -\frac{\eta^k}{k} n^n\right).$$

Integrating over $\eta \in (0,\infty)$ and multiplying by η^{z-1}, we obtain (28) after a simple change of variables. □

Next, we state a result dealing with the lower bound of $\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta)$.

Theorem 2.4 *Let $a_n > 0$, $b_n - a_n > 0$, $\eta > 0$, $\zeta \geq 0$ and $\phi \in (0,\infty)$. Then,*

$$\frac{\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta)}{\beta_k(\phi;\eta)} \geq {}_1F_{1,k}\left(b_n - a_n; b; \frac{\eta^k n^n}{k}\right) \exp\left(-\frac{\zeta^k}{\eta^k n^n}\right), \tag{29}$$

where $\beta_k(\phi;\eta)$ is defined in (9).

Proof Using (9), we have

$$\begin{aligned}\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta) = \frac{1}{k^{n-1}} \int_{E_{n-1}} \prod_{i=1}^n t_i^{\frac{\phi_i}{k}-1} \exp\left(-\frac{\eta^k}{k\pi(t)} - \zeta^k \frac{\pi(t)}{\eta^k}\right) \\ \cdot {}_1F_{1,k}\left(b_n - a_n; b_n; \frac{\eta^k}{k\pi(t)} + \zeta^k \frac{\pi(t)}{\eta^k}\right) dt,\end{aligned}$$

which can be rewritten as

$$\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta) = \frac{1}{k^{n-1}} \int_{E_{n-1}} \left(\prod_{i=1}^{n} t_i^{\frac{\phi_i}{k}-1} e^{-\frac{\eta^k}{k\pi(t)}} \right) e^{-\frac{\zeta^k \pi(t)}{\eta^k}}$$

$$\cdot {}_1F_{1,k}\left(b_n - a_n; b_n; \frac{\eta^k}{k\pi(t)} + \zeta^k \frac{\pi(t)}{\eta^k} \right) dt. \tag{30}$$

Using Lemma 2.2, we have $e^{-\frac{\zeta^k \pi(t)}{\eta^k}} \geq e^{-\frac{\zeta^k}{\eta^k n^n}}$ and $\frac{\eta^k}{k\pi(t)} + \zeta^k \frac{\pi(t)}{\eta^k} \geq \frac{\eta^k}{k} n^n$. Using this in (30) and with the help of (9), we have (29). □

Corollary 2.2 *Under the assumptions of Theorem 2.4, for any $z > 0$, we have*

$$\int_0^\infty r^{z-1} \beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta) dr \geq \eta^z \left(\frac{n^n}{k} \right)^{\frac{z}{k}} \Gamma_k(z) \beta_k(\phi;\eta)\, {}_1F_{1,k}\left(b_n - a_n; b_n; \frac{\eta^k k^n}{n} \right). \tag{31}$$

If $z = 1$, then

$$\int_0^\infty \beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta) dr \geq \eta^z \left(\frac{n^n}{k} \right)^{\frac{z}{k}} \beta_k(\phi;\eta) \Gamma_k(1)\, {}_1F_{1,k}\left(a_n - b_n; b_n; \frac{\eta^k k^n}{n} \right). \tag{32}$$

Proof Multiplying (29) by r^{z-1} and integrating over $r \in (0,\infty)$, we have

$$\int_0^\infty r^{z-1} \beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta) dr$$

$$\geq \beta_k(\phi;\eta)\, {}_1F_{1,k}\left(a_n - b_n; b_n; \frac{\eta^k}{k} n^n \right) \int_0^\infty r^{z-1} \exp\left(-\frac{\zeta^k}{\eta^k n^n} \right) dr.$$

Setting $t = \left(\frac{kr^k}{\eta^k n^n} \right)^{\frac{1}{k}}$ and making simplification, we obtain (31) and then (32). □

Our next result is as follows.

Theorem 2.5 *Let $a_n > 0$, $b_n - a_n > 0$, $\eta > 0$, $\zeta \geq 0$ and $\phi \in (0,\infty)$. Then,*

$$\frac{\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta)}{\beta_k(\phi;\eta)} \geq {}_1F_{1,k}\left(a_n; b_n; -\frac{\zeta^k}{\eta^k n^n} \right). \tag{33}$$

Proof Using (6), we have

$${}_1F_{1,k}\left(a_n; b_n; -\frac{\eta^k}{k\pi(t)} - \zeta^k \frac{\pi(t)}{\eta^k} \right) = \frac{\Gamma(b_n)}{k\Gamma(a_n)\Gamma(b_n - a_n)}$$

$$\cdot \int_0^1 b^{\frac{a_n}{k}-1} (1-b)^{\frac{b_n-a_n}{k}-1} \exp\left(-b \frac{\eta^k}{k\pi(t)} \right) \exp\left(-b\zeta^k \frac{\pi(t)}{\eta^k} \right) db.$$

Using Lemma 2.2,

$$\exp\left(-b\frac{\eta^k}{k\pi(t)}\right) \geq \exp\left(-\frac{\eta^k}{k\pi(t)}\right)$$

and

$$\exp\left(-b\zeta^k\frac{\pi(t)}{\eta^k}\right) \geq \exp\left(-b\frac{\zeta^k}{\eta^k n^n}\right).$$

Applying these in

$$\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta) = \frac{1}{k^{n-1}}\int\limits_{E_{n-1}}\prod_{i=1}^{n} t_i^{\frac{\phi_i}{k}-1}\,{}_1F_{1,k}\left(a_n;b_n;-\frac{\eta^k}{k\pi(t)}-\zeta^k\frac{\pi(t)}{\eta^k}\right)dt,$$

and using the uniform convergence of the involved integrals, we have

$$\begin{aligned}\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta) \geq{}& \frac{1}{k^{n-1}}\frac{\Gamma(b_n)}{k\Gamma(a_n)\Gamma(b_n-a_n)}\\ &\cdot\int\limits_0^1 b^{\frac{a_n}{k}-1}(1-b)^{\frac{b_n-a_n}{k}-1}\exp\left(-b\frac{\zeta^k}{\eta^k n^n}\right)db\\ &\cdot\int\limits_{E_{n-1}}\prod_{i=1}^{n} t_i^{\frac{\phi_i}{k}-1}\exp\left(-\frac{\eta^k}{k\pi(t)}\right)dt.\end{aligned}$$

Hence, (33). □

Corollary 2.3 *Under the assumptions of Theorem 2.4, for any $z>0$, we have*

$$\int\limits_0^\infty \zeta^{z-1}\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta)d\zeta \geq \eta^z\left(\frac{n^n}{k}\right)^{\frac{z}{k}}\beta_k(\phi;\eta)\Gamma_k^{(a_n,b_n)}(z),$$

where $\Gamma_k^{(c,d)}(z)$ is defined in (13). If $z=1$, then

$$\int\limits_0^\infty \beta_{\zeta,k}^{(a_n,b_n)}d\zeta \geq \eta\left(\frac{n^n}{k}\right)^{\frac{1}{k}}\beta_k(\phi;\eta)\Gamma_k^{(a_n,b_n)}(1).$$

Proof Multiplying (33) by ν^{z-1} and integrating over $\nu\in(0,\infty)$, we obtain

$$\int\limits_0^\infty \zeta^{z-1}\beta_{\zeta,k}^{(a_n,b_n)}(\phi;\eta)d\zeta \geq \beta_k(\phi;\eta)\int\limits_0^\infty \zeta^{z-1}\,{}_1F_{1,k}\left(a_n;b_n;-\frac{\zeta^k}{\eta^k n^n}\right)d\nu.$$

Setting $\zeta=\eta t\frac{(n^n)^{\frac{1}{k}}}{k^{\frac{1}{k}}}$ and simplifying yields the desired result. □

2.2 Generalized beta *k* function of the second kind

In this section, we discuss the generalized k-beta function of the second kind. Before describing its representation, we introduce several notations used throughout this section. Let $\phi = (\phi_1, \dots, \phi_n)$, $\eta = (\eta_1, \dots, \eta_n)$, $p = (p_1, \dots, p_n)$, $q = (q_1, \dots, q_n)$ and $\zeta = (\zeta_1, \dots, \zeta_n)$.

Definition 2.2 Let $\phi, p, q, \eta \in (0, \infty)^n$ and $\zeta \in [0, \infty)^n$, we define a generalized k-beta function as

$$\beta_{\zeta,k}^{(p,q)}(\phi;\eta) =: \frac{1}{k^{n-1}} \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{\frac{\phi_i}{k}-1} {}_1F_{1,k}\left(p_i; q_i; -\frac{\eta_i^k}{kt_i} - \zeta_i^k \frac{t_i}{f_i^k}\right) dt, \tag{34}$$

where $dt = (dt_1, \dots, dt_{n-1})$ and $t_n = 1 - \sum_{i=1}^{n-1} t_i$. If $\zeta = 0$, then

$$\beta^{(p,q)}(\phi;\eta) =: \frac{1}{k^{n-1}} \int_{E_{n-1}} \prod_{i=1}^{n} t_i^{\frac{\phi_i}{k}-1} {}_1F_{1,k}\left(p_i; q_i; -\frac{\eta_i{}^k}{kt_i}\right) dt. \tag{35}$$

If $p = q$ and $n = 2$, then (35) becomes similar to (1). If $p = q$ and $\zeta = 0$, then (34) is exactly (10).

Remark 2.1 (34) is well-defined because due to the uniform convergence of the stated series in (5), we can interchange series and integral defined in (34). Further, such integrals are uniformly convergent in any compact set included in the interior of E_{n-1}. This allows for differentiation and limit under the integral sign of (34). We may state $\lim_{\zeta \to 0} \beta_{\zeta,k}^{(p,q)}(\phi;\eta) = \beta_k^{(p,q)}(\phi;\eta)$.

Proposition 2.3 *For any $p, \phi, \eta \in (0, \infty)^n$, we have*

$$\beta_k(\phi;\eta) = \lim_{\zeta \to 0} \beta_{\zeta,k}^{(p;p)}(\phi;\eta) =: \beta_k^{(p,p)}(\phi;\eta).$$

Proof Using (5) and (35), we have for $i = 1, 2, \dots, n$

$${}_1F_{1,k}\left(p_i; p_i; -\frac{\eta_i^k}{kt_i}\right) = \sum_{i=1}^{\infty} \frac{\left(-\frac{\eta_i^k}{kt_i}\right)^m}{m!} = e^{-\frac{\eta_i^k}{kt_i}}.$$

Combining (10) and (34) gives the required result. □

Next, we mention some inequalities involving the described beta function. Our first result is related to convexity of $\beta_{\zeta,k}^{(p,q)}(\phi;\eta)$.

Theorem 2.6 *Let $\eta, p, q - p \in (0, \infty)$ and $\zeta \in [0, \infty)$. Then,*

$$\left(\beta_{\zeta,k}^{(p,q)}(\phi + \psi;\eta)\right)^2 \le \beta_{\zeta,k}^{(p,q)}(2\phi;\eta)\beta_{\zeta,k}^{(p,q)}(2\psi;\eta)$$

holds for any $\phi, \psi \in (0, \infty)^n$. Therefore, $\beta_{\zeta,k}^{(p,q)}(\phi;\eta)$ is convex on $(0, \infty)^n$.

Proof The proof is similar to that of Theorem 2.1. □

Theorem 2.7 *We have*

$$\frac{\beta^{(p,q)}_{\zeta,k}(\phi;\eta)}{\beta_k(l)} \leq \prod_{i=1}^{n} e^{-\frac{\eta_i^k}{k}}\ {}_1F_{1,k}\left(q_i-p_i;q_i;\frac{\eta_i^k}{k}+\frac{\zeta_i^k}{\eta_i^k}\right),$$

where $\beta_k(\phi)$ *is defined in* (8).

Proof Since the real valued function $l \mapsto {}_1F_{1,k}(c_n;d_n;l)$ is increasing and $0<t_i\leq 1$, $i=1,\ldots,n$, we have

$$\beta^{(p,q)}_{\zeta,k}(\phi;\eta) \leq \frac{1}{k^{n-1}}\int\limits_{E_{n-1}} \prod_{i=1}^{n} t_i^{\frac{\phi_i}{k}-1}\ {}_1F_{1,k}\left(p_i;q_i;-\frac{\eta_i^k}{k}-\zeta_i^k\frac{t_i}{\eta_i^k}\right)dt.$$

Using (7) gives

$$\begin{aligned}\beta^{(p,q)}_{\zeta,k}(\phi;\eta) \leq{}& \frac{1}{k^{n-1}}\int\limits_{E_{n-1}} \prod_{i=1}^{n} t_i^{\frac{\phi_i}{k}-1}\exp\left(-\frac{\eta_i^k}{kt_i}-\frac{\zeta^k t_i}{\eta_i^k}\right)\\ &\cdot{}_1F_{1,k}\left(q_i-p_i;q_i;\frac{\eta_i^k}{k}+\zeta_i^k\frac{t_i}{\eta_i}^k\right)dt. \qquad (36)\end{aligned}$$

It is clear that $\exp\left(\frac{-\eta_i^k}{k}\right)-\zeta_i^k\frac{t_i}{\eta_i^k}\leq e^{\frac{-\eta_i^k}{k}}$ and $\frac{\eta_i^k}{k}+\zeta_i^k\frac{t_i}{\eta_i^k}\leq\frac{\eta_i^k}{k}+\frac{\zeta_i^k}{\eta_i^k}$. Using these in (36), we obtain the desired result. □

Now, we provide a lower bond of the described function.

Theorem 2.8 *We have*

$$\frac{\beta^{(p,q)}_{\zeta,k}(\phi;\eta)}{\beta_k(\phi;\eta)} \geq \prod_{i=1}^{n} e^{-\frac{\zeta_i^k}{\eta_i^k}}\ {}_1F_{1,k}\left(q_i-p_i;q_i;m_i\right), \qquad (37)$$

where $\beta(\phi;\eta)$ *is defined in* (10) *and* $m_i=\max\left(2\sqrt{\zeta_i^k},\eta_i^k\right)$. *If* $\eta_i^k\geq\sqrt{\zeta_i^k}$, $i=1,\ldots,n$, *then* (37) *can be refined as*

$$\frac{\beta^{(p,q)}_{\zeta,k}(\phi;\eta)}{\beta_k(\phi;\eta)} \geq \prod_{i=1}^{n} e^{-\frac{\zeta_i^k}{\eta_i^k}}\ {}_1F_{1,k}\left(q_i-p_i;q_i;\frac{\eta_i^k}{k}+\frac{\zeta_i^k}{\eta_i^k}\right). \qquad (38)$$

Proof Using (7) and (34), we obtain

$$\begin{aligned}\beta^{(p,q)}_{\zeta,k}(\phi;\eta) ={}& \frac{1}{k^{n-1}}\int\limits_{E_{n-1}}\left(\prod_{i=1}^{n} t_i^{\frac{\eta_i^k}{k}-1}e^{-\frac{\eta_i^k}{kt_i}}\right)e^{-\zeta_i^k\frac{t_i}{\eta_i^k}}\\ &\cdot{}_1F_{1,k}\left(q_i-p_i;q_i;\frac{\eta_i^k}{kt_i}+\zeta_i^k\frac{t_i}{\eta_i^k}\right)dt. \qquad (39)\end{aligned}$$

It is easy to see that $\frac{\eta_i^k}{kt_i} + \zeta_i^k \frac{t_i}{\eta_i^k} \geq 2\sqrt{\zeta_i^k}$, further $0 < t_i < 1$, and $h \mapsto {}_1F_1(a_n; b_n; c)$ is also increasing. We have $e^{-\zeta_i^k \frac{t_i}{\eta_i^k}} \geq e^{-\frac{\zeta_i^k}{\eta_i^k}}$ and

$$ {}_1F_{1,k}\left(q_i - p_i; q_i; \frac{\eta_i^k}{kt_i} + \zeta_i^k \frac{t_i}{\eta_i^k}\right) \geq {}_1F_{1,k}\left(q_i - p_i; q_i; \max\left(2\sqrt{\zeta_i^k}, \eta_i^k\right)\right). $$

Substituting into (39) and using (10), we have (37). (η, ζ) with $\eta \geq \sqrt{\zeta^k}$ implies $\inf\left(\frac{\eta^k}{t} + \zeta^k \frac{t}{\eta^k}\right) = \eta^k + \frac{\zeta^k}{\eta^k}$, so (38) refines to (37). The proof is complete. □

We have the next result.

Theorem 2.9 *Let $\phi, \eta, p, s - p \in (0, \infty)^n$. For any $z =: (z_1, \ldots, z_n)$, we have*

$$ \begin{aligned} &\int_{(0,\infty)^n} \prod_{i=1}^n \zeta_i^{z_i - 1} \beta_{\zeta,k}^{(p,q)}(\phi; \eta) dg \\ &\geq \beta_k(\phi, \eta)\Gamma_k(z)\left(\prod_{i=1}^n \eta_i^{z_i} \frac{1}{k^{\frac{z_i}{k}}}\right) \prod_{i=1}^n {}_1F_{1,k}\left(q_i - p_i; q_i; f_i^k\right), \end{aligned} \tag{40} $$

where $d\zeta =: d\zeta_1 \cdots d\zeta_n$. If $z = e = (1, \ldots, 1)$, then

$$ \int_{(0,\infty)^n} \beta_{\zeta,k}^{(p,q)}(\phi; \eta) dg \geq \beta_k(\phi, \eta)\left(\prod_{i=1}^n \eta_i^{z_i} \frac{1}{k^{\frac{z_i}{k}}}\right) \prod_{i=1}^n {}_1F_{1,k}\left(q_i - p_i; q_i; f_i^k\right). \tag{41} $$

Proof Multiply (37) by $\prod_{i=1}^n \zeta_i^{z_i - 1}$ and integrate over $\zeta \in (0, \infty)^n$ to obtain

$$ \begin{aligned} &\int_{(0,\infty)^n} \prod_{i=1}^n \zeta_i^{z_i - 1} \beta_{\zeta,k}^{(p,q)}(\phi; \eta) dg \\ &\geq \beta_k(\phi; \eta)\, {}_1F_{1,k}\left(q_i - p_i; q_i; \eta_i^k\right) \int_{(0,\infty)^n} \prod_{i=1}^n \zeta_i^{z_i - 1} e^{-\frac{\zeta_i^k}{\eta_i^k}} dg. \end{aligned} $$

Setting $t = \left(\frac{k\zeta_i^k}{\eta_i^k}\right)^{\frac{1}{k}}$, $i = 1, \ldots, n$, we have

$$ \begin{aligned} &\int_{(0,\infty)^n} \prod_{i=1}^n \zeta_i^{z_i - 1} \beta_{\zeta,k}^{(p,q)}(\phi; \eta) dg \\ &\geq \beta_k(\phi; \eta) \prod_{i=1}^n \eta_i^z \frac{1}{k^{\frac{z}{k}}}\, {}_1F_{1,k}\left(q_i - p_i; q_i; \eta_i^k\right) \int_{(0,\infty)^n} (t_i)^{z_i - 1} e^{-\frac{t_i^k}{k}} dt. \end{aligned} $$

Hence (40). Taking $z_i = 1$, $i = 1, \ldots, n$ in (40), we have (41). The proof is complete. □

The following result may also be stated.

Theorem 2.10 *Let $\phi, \eta, p, q-p \in (0,\infty)^n$ and $\zeta \in [0,\infty)$. Then,*

$$\frac{\beta_{\zeta,k}^{(p,q)}(\phi;\eta)}{\beta_k(\phi;\eta)} \geq \prod_{i=1}^{n} {}_1F_{1,k}\left(p_i;q_i;-\frac{\zeta_i^k}{\eta_i^k}\right). \tag{42}$$

Proof Using (6), we have

$$\begin{aligned}{}_1F_{1,k}\left(p_i;q_i;-\frac{\eta_i^k}{kt_i}-\zeta_i^k\frac{t_i}{\eta_i^k}\right) &= \frac{\Gamma(q_i)}{k\Gamma(p_i)-\Gamma(q_i-p_i)} \\ &\cdot \int_0^1 u^{\frac{\eta_i}{k}-1}(1-u)^{\left(\frac{q_i-p_i}{k}-1\right)} e^{-u\frac{\eta_i^k}{kt_i}} e^{-u\zeta_i^k\frac{t_i}{\eta_i^k}}\,du. \end{aligned} \tag{43}$$

Since $0<t_i\leq 1$, $i=1,\ldots,n$ and $u\in[0,1]$, we have $e^{-u\frac{\eta_i^k}{kt_i}} \geq e^{-\frac{\eta_i^k}{kt_i}}$ and $e^{-\zeta_i^k u\frac{t_i}{\eta_i^k}} \geq e^{-u\frac{\zeta_i^k}{\eta_i^k}}$. Using this in (43) and then in (34), we have

$$\begin{aligned}\beta_{\zeta,k}^{(p,q)}(\phi;\eta) &\geq \frac{1}{k^{n-1}}\int_{E_{n-1}}\prod_{i=1}^{n} t_i^{\frac{\eta_i}{k}-1} e^{-\frac{\eta_i^k}{kt_i}}\,dt \prod_{i=1}^{n}\frac{\Gamma(q_i)}{k\Gamma(p_i)-\Gamma(q_i-p_i)} \\ &\cdot \int_0^1 u^{\frac{r_i}{k}-1}(1-u)^{\frac{q_i-p_i}{k}-1} e^{-u\frac{\zeta_i^k}{\eta_i^k}}\,du.\end{aligned}$$

This with (10) and (6) yields (42). □

We now turn to our final result.

Theorem 2.11 *Let $\phi, \eta, q, p-q \in (0,\infty)^n$. For any $z =: (z_1,\ldots,z_n) \in (0,\infty)^n$, we have*

$$\int_{(0,\infty)^n}\prod_{i=1}^{n}\zeta_i^{z_i-1}\beta_{\zeta,k}^{(p,q)}(\phi;\eta)\,dg \geq \beta_k(\phi;\eta)\Gamma_k^{(p,q)}(z)\prod_{i=1}^{n}\left(\frac{\eta_i^{z_i}}{k^{\frac{z_i}{k}}}\right).$$

If $z=e=:(1,\ldots,1)$, then

$$\int_{(0,\infty)^n}\beta_{\zeta,k}^{(p,q)}(\phi;\eta)\,dg \geq \beta_k(\phi;\eta)\Gamma_k^{(p,q)}(e)\prod_{i=1}^{n}\eta_i.$$

Proof Similar to the proof of Theorem 2.10. Using (42), we obtain

$$\beta_{\zeta,k}^{(p,q)}(\phi;\eta) \geq \beta_k(\phi;\eta)\prod_{i=1}^{n} {}_1F_{1,k}\left(p_i;q_i;-\frac{\zeta_i^k}{\eta_i^k}\right).$$

Multiplying by $\zeta_i^{z_i-1}$ and integrating over $g\in(0,\infty)$, we obtain our required result. □

3 Conclusions

In this note, we presented generalized inequalities involving beta and gamma functions and their generalizations. Basic representations of beta and gamma functions were included, along with representations involving confluent hypergeometric functions. Some basic relationships between gamma and beta functions have been provided, and refined inequalities for the extended beta function were introduced. We have also discussed upper and lower bounds for an extended beta function.

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