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Control Functions in G-Metric Spaces: Novel Methods for θ-Fixed Points and θ-Fixed Circles with an Application

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Abstract: The purpose of this paper is to present some new contraction mappings via control functions. In addition, some fixed point results for $(\Theta, \alpha, \theta, \Psi)$ contraction, rational $(\Theta, \alpha, \theta, \Psi)$ contraction and almost $(\Theta, \alpha, \theta, \Psi)$ contraction mappings are obtained. Moreover, under contraction mappings of types (I), (II), and (III) of $(\Theta, \theta, \Psi)_{v_0}$, several fixed circle solutions are provided in the setting of a G-Metric space. Our results extend, unify, and generalize many previously published papers in this direction. In addition, some examples to show the reliability of our results are presented. Finally, a supporting application that discusses the possibility of a solution to a nonlinear integral equation is incorporated.

Keywords: θ-fixed point; θ-fixed circle; nonlinear integral equation; almost $(\Theta, \alpha, \theta, \Psi)$ contraction; a $(\Theta, \theta, \Psi)_{v_0}$-type contraction

MSC: 47H9; 46B80; 47H10

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1. Introduction

Mustafa and Sims [1] proposed G-Metric space (GMS for short) to extend and generalize the notion of metric space. The Banach contraction mapping [2] was generalized by the authors of this paper in the context of a GMS. Following this initial report, a number of authors defined many well-known fixed-point theorems in GMS (see, e.g., [2–4]). There is a close relation between a regular metric space and a GMS, since one is adapted from the other. For more details, see [5–7].

In fact, the nature of a GMS is to comprehend the geometry of three points rather than two points via a triangle's perimeter. However, these aspects were not given significant weight in the majority of the published articles dealing with a GMS. As a result, the vast majority of results were achieved by translating the contraction conditions from the setting of metric space to a GMS without sufficiently incorporating the peculiarities of the GMS. Several fixed point (FP) theorems in the literature that are used in the context of a GMS can be deduced from some existing results when used in the context of a (quasi-)metric space, according to Samet et al. [8] and Jleli-Samet [9]. In fact, one can establish an equivalent FP theorem in usual metric space if the contraction condition of the FP theorem on a GMS can be reduced to two variables instead of three variables.

In contrast to the F contractions suggested by Wardowski [10], Jleli and Samet [11] introduced φ contractions in 2014. They represented the entire family of functions φ by Θ^* and proved some FP theorems for such contractions in usual metric spaces. In the subsequent works, Jleli and Samet [11] and Liu et al. [12] presented the ideas of φ-type Suzuki contractions and φ-type contractions, found some new FP theorems in complete metric spaces, and solved nonlinear Hammerstein integral equations.

Jleli et al. [13] obtained a number of θ-FP results based on the notion of new control functions, and they also presented the novel ideas of θ-FP and θ-Picard mappings. In addition, they asserted that certain FP outcomes in partial metric spaces can be deduced from these θ-FP results in metric spaces. Many well-known results have been released after the notions of Jleli et al. [13] such as the definition of (F, φ, θ) contractions [14], $(F, \varphi, \alpha - \psi)$ contractions [15], and $(F, \varphi, \alpha - \psi)$-weak contractions by the control function in [16], which improved upon the consequences of Kumrod and Sintunavarat [17]. They solved the existence of solutions for boundary value problems in second-order ordinary differential equations. For more details, see [18–20].

Recently, many different elements of the geometric features of non-unique FPs have been thoroughly researched; examples include the fixed-circle problem and the fixed-disc problem. As a new method of generalizing the FP theorem, Özgür and Taş [21] developed the fixed circle problem in a metric space and the concept of a fixed circle. We encourage readers to [22–27] for some recent research on the fixed-circle and fixed-disc problems.

Since the writers did not address this direction in the GMS and similar to previous works, in this article, a number of new contractions with control functions are established. Our findings enhance and expand upon some earlier FP results. We also provide several examples and an application to demonstrate the usefulness of our findings.

2. Preliminaries

In this section, we go over several fundamental ideas and known facts. The symbols $\mathbb{R}$, $\mathbb{R}^+$, $\mathbb{N}$, $\Lambda(\Im)$ and Q_θ refer to the set of all real numbers, non-negative real numbers, natural numbers, FPs and zero points of θ, respectively.

Definition 1 ([1])**.** *Let Ω be a non-empty set and $G : \Omega^3 \to \mathbb{R}^+$ be a function fulfilling the properties below for all $v, \varrho, \sigma, s \in \Omega$*

(g_1) $G(v, \varrho, \sigma) = 0$ *if* $v = \varrho = \sigma$;
(g_2) $0 < G(v, v, \varrho)$ *with* $v \neq \varrho$;
(g_3) $G(v, v, \varrho) \leq G(v, \varrho, \sigma)$ *with* $\varrho \neq \sigma$;
(g_4) $G(v, \varrho, \sigma) = G(v, \sigma, \varrho) = G(\varrho, \sigma, v) = \ldots$ *(symmetry in all three variables);*
(g_5) $G(v, \varrho, \sigma) \leq G(v, s, s) + G(s, \varrho, \sigma)$ *(rectangle inequality).*

Here, the function G is called a G-Metric on Ω and the pair (Ω, G) is called a GMS.

Note, each G-Metric on Ω establishes the metric d_G on Ω by

$$d_G(v, \varrho) = G(v, \varrho, \varrho) + G(\varrho, v, v), \ \forall v, \varrho \in \Omega.$$

Example 1 ([1])**.** *Let (Ω, G) be a GMS. The function $G : \Omega^3 \to \mathbb{R}^+$ described as*

$$G(v, \varrho, \sigma) = \max\{d(v, \varrho), d(\varrho, \sigma), d(v, \sigma)\},$$

or

$$G(v, \varrho, \sigma) = d(v, \varrho) + d(\varrho, \sigma) + d(v, \sigma),$$

for all $v, \varrho, \sigma \in \Omega$ is a G-Metric on Ω.

Definition 2 ([1])**.** *Let (Ω, G) be a GMS and $\{v_k\}$ be a sequence of points of Ω. Then, $\{v_k\}$ is called:*

(i) *A G-convergent to $v \in \Omega$ if*

$$\lim_{k,l \to \infty} G(v, v_k, v_l) = 0,$$

that is, for any $\epsilon > 0$, there is $K \in \mathbb{N}$ so that $G(v, v_k, v_l) < \epsilon$ for all $k, l \geq K$. Moreover, v is called the limit of the sequence and write $\lim_{k \to \infty} v_k = v$ or $v_k \to v$.

(ii) *A G-Cauchy sequence if for any $\epsilon > 0$, there exists $K \in \mathbb{N}$ so that $G(v_n, v_k, v_l) < \epsilon$ for all $n, k, l \geq K$, that is $G(v_n, v_k, v_l) \to 0$ as $n, k, l \to \infty$.*

Definition 3 ([1]). *If every G-Cauchy sequence is G-convergent in a GMS (Ω, G), the space (Ω, G) is said to be G-complete.*

Proposition 1 ([1]). *Let (Ω, G) be a GMS. The statments below are equivalent:*

(i) $\{v_k\}$ *is G-convergent to* v;
(ii) $G(v_k, v_k, v) \to 0$ *as* $k \to \infty$;
(iii) $G(v_k, v, v) \to 0$ *as* $k \to \infty$;
(iv) $G(v, v_k, v_l) \to 0$ *as* $k, l \to \infty$.

Proposition 2 ([1]). *Let (Ω, G) be a GMS. The statements below are equivalent:*

(a) $\{v_k\}$ *is a G-Cauchy sequence;*
(b) *For any* $\epsilon > 0$, *there is* $K \in \mathbb{N}$ *so that* $G(v_n, v_k, v_k) < \epsilon$ *for all* $k, l \geq K$.

Lemma 1 ([1]). *Let (Ω, G) be a GMS. Then, the inequality below holds*

$$G(v, v, \varrho) \leq 2G(v, \varrho, \varrho), \text{ for all } v, \varrho \in \Omega.$$

Definition 4 ([1]). *Let $\Im$ be a self-mapping defined on a GMS (Ω, G). Then, $\Im$ is called G-continuous if $\{\Im v_k\}$ is G-convergent to $\Im v$ whenever $v_k \to v$ as $k \to \infty$.*

Let Θ^* be the family of all functions $\Theta : (0, \infty) \to (0, \infty)$ fulfilling the conditions below:

- Θ is continuous;
- For all positive sequences $\{\tau_k\}$, $\lim_{k\to\infty} \Theta(\tau_k) = 0$ if $\lim_{k\to\infty} \tau_k = 0$;
- Θ is nondecreasing.

Let Ψ^* be the family of all functions $\Psi : (0, \infty) \to (0, \infty)$ fulfilling the conditions below:

- Ψ is nondecreasing;
- $\lim_{k\to\infty} \Psi^k(\tau) = 0$ for $\tau > 0$, where Ψ^k stands for the k-th iterate of Ψ.

Remark 1 ([28]). *If $\Psi \in \Psi^*$, then $\Psi(\tau) < \tau$, for all $\tau > 0$.*

Theorem 1 ([29]). *Let (Ω, d) be a CMS and $\Im : \Omega \to \Omega$ be a mapping so that*

$$d(\Im v, \Im \varrho) \geq 0 \text{ implies } \Theta(d(\Im v, \Im \varrho)) \leq \Psi(d(v, \varrho)),$$

for all $v, \varrho \in \Omega$, where $\Theta, \Psi : (0, \infty) \to \mathbb{R}$ satisfies the assertions below:

(T_1) *The function* Θ *is nondecreasing;*
(T_2) *For all* $\tau > 0$, $\Psi(\tau) < \Theta(\tau)$;
(T_3) *For all* $\epsilon > 0$, $\limsup_{\tau\to\epsilon} \Psi(\tau) < \Theta(\epsilon)$.

Then, $\Im$ has a unique FP.

Remark 2. *If $\rho(\tau) = p(\tau)$, $\lambda(\tau) = q(p(\tau))$. If $q \in \Psi$, $p \in \Theta$ and q is continuous, then the assertions below are true:*

(1) ρ *is a nondecreasing function;*
(2) $\lambda(\tau) < \rho(\tau)$, *for all* $\tau > 0$;
(3) *For all* $\epsilon > 0$, $\limsup_{\tau\to\epsilon} \lambda(\tau) < \rho(\epsilon)$.

Definition 5 ([13]). *Assume that $\theta : \Omega \to [0, \infty)$ and $\Im : \Omega \to \Omega$. A point $v \in \Omega$ is said to be a θ-FP of $\Im$ if $\Im v = v$ and $\theta(v) = 0$.*

Definition 6 ([13]). *Assume that $\theta : \Omega \to [0, \infty)$. A mapping $\Im : \Omega \to \Omega$ is said to be a θ-Picard mapping if the assertions below hold:*

(i) $Q_\theta \cap \Lambda(\Im) = \{v\}$;
(ii) *For* $v_0 \in \Omega$, $\lim_{k\to\infty} \Im^k v_0 = v$.

A new control function is presented by Jleli et al. [13] as follows: Assume that $\widetilde{\alpha} : [0,\infty)^3 \to [0,\infty)$ verifying the following axioms:

(a) $\max\{\ell_1, \ell_2\} \le \widetilde{\alpha}(\ell_1, \ell_2, \ell_3)$;
(b) $\widetilde{\alpha}(0,0,0) = 0$;
(c) $\widetilde{\alpha}$ is continuous.

We denote all control functions $\widetilde{\alpha}$ by α^*.

Example 2 ([13])**.** *Let* $\widetilde{\alpha}_1(\ell_1,\ell_2,\ell_3) = \ell_1 + \ell_2 + \ell_3$, $\widetilde{\alpha}_2(\ell_1,\ell_2,\ell_3) = \max\{\ell_1,\ell_2\} + \ell_3$ *and* $\widetilde{\alpha}_3(\ell_1,\ell_2,\ell_3) = \ell_1 + \ell_1^2 + \ell_2 + \ell_3$ *for* $\ell_1,\ell_2,\ell_3 \in [0,\infty)$*. Then* $\widetilde{\alpha}_1, \widetilde{\alpha}_2, \widetilde{\alpha}_3 \in \alpha^*$.

In addition, on a CMS, they proved the following theorem:

Theorem 2 ([13])**.** *Let* (Ω, d) *be a CMS and* $\Im : \Omega \to \Omega$ *be a mapping so that*

$$\alpha(d(\Im v, \Im \varrho), \theta(\Im v), \theta(\Im \varrho)) \le \delta \alpha(d(v,\varrho), \theta(v), \theta(\varrho)),$$

where the function θ *is a lower semicontinuous and* $\delta \in (0,1)$*. Then,* $\Im$ *is a* θ*-Picard mapping.*

Definition 7 ([30])**.** *Let* $\Im : \Omega \to \Omega$ *be a self-mapping defined on a metric space* (Ω, d) *and* $\theta : \Omega \to [0,\infty)$*. Define* $s = \inf\{d(v, \Im v) : v \neq \Im v\}$*. Then*

(i) *A circle* $C_{v_0,s} = \{v \in \Omega : d(v, v_0) = s\}$ *in* Ω *is called a* θ*-fixed circle of* $\Im$ *if* $C_{v_0,s} \subset \Lambda(\Im) \cap Q_\theta$.
(ii) *A disc* $D_{v_0,s} = \{v \in \Omega : d(v, v_0) \le s\}$ *in* Ω *is called a* θ*-fixed disc of* $\Im$ *if* $D_{v_0,s} \subset \Lambda(\Im) \cap Q_\theta$.

3. θ-Fixed Point Theorems

In this part, we present some novel contractions and some corresponding findings.

According to the control function defined on [13], we can define another control function in line with our results as follows:

Definition 8. *Let* $\alpha : [0,\infty)^4 \to [0,\infty)$ *be a function satisfying*

(α_1) $\max\{\ell_1,\ell_2,\ell_3\} \le \alpha(\ell_1,\ell_2,\ell_3,\ell_4)$;
(α_2) $\alpha(0,0,0,0) = 0$;
(α_3) α *is continuous.*

We denote all control functions α by $\widehat{\alpha}$.

Example 3. *Let* $\alpha_1(\ell_1,\ell_2,\ell_3,\ell_4) = \ell_1+\ell_2+\ell_3+\ell_4$, $\alpha_2(\ell_1,\ell_2,\ell_3,\ell_4) = \max\{\ell_1,\ell_2,\ell_3\} + \ell_4$, $\alpha_3(\ell_1,\ell_2,\ell_3,\ell_4) = \ell_1 + \ell_1^2 + \ell_2 + \ell_3 + \ell_4$ *and* $\alpha_4(\ell_1,\ell_2,\ell_3,\ell_4) = \max\{\ell_1,\ell_2\} + \max\{\ell_3,\ell_4\}$ *for* $\ell_1,\ell_2,\ell_3,\ell_4 \in [0,\infty)$*. Then,* $\alpha_1,\alpha_2,\alpha_3,\alpha_4 \in \widehat{\alpha}$.

Now, we can present our results in this part. We begin with the following definition:

Definition 9. *We say that a mapping* $\Im : \Omega \to \Omega$ *is a* $(\Theta, \alpha, \theta, \Psi)$ *contraction in a GMS* (Ω, G) *if there are* Θ *and* Ψ *verifying axioms* $(T_1) - (T_3)$ *of Theorem 1 and the inequality*

$$\Theta(\alpha(G(\Im v, \Im\varrho, \Im\sigma), \theta(\Im v), \theta(\Im\varrho), \theta(\Im\sigma))) \le \Psi(\alpha(G(v,\varrho,\sigma), \theta(v), \theta(\varrho), \theta(\sigma))), \quad (1)$$

for all $v, \varrho, \sigma \in \Omega$ *so that* $\alpha(G(\Im v, \Im\varrho, \Im\sigma), \theta(\Im v), \theta(\Im\varrho), \theta(\Im\sigma)) > 0$.

Theorem 3. *Let* (Ω, G) *be a complete GMS and* $\Im : \Omega \to \Omega$ *be a* $(\Theta, \alpha, \theta, \Psi)$ *contraction mapping. Then,* $\Im$ *is a* θ*-Picard mapping provided that* θ *is lower semicontinuous (lsc, for short).*

Proof. At first, we show that $\Lambda(\Im) \subseteq Q_\theta$. Let there be $v \in \Lambda(\Im)$ so that $\theta(v) \neq 0$. Put $v = \varrho = \sigma$ in (1); then, we have

$$\begin{aligned}\Theta(\alpha(0,\theta(v),\theta(v),\theta(v))) &= \Theta(\alpha(G(\Im v,\Im v,\Im v),\theta(\Im v),\theta(\Im v),\theta(\Im v)))\\ &\leq \Psi(\alpha(G(v,v,v),\theta(v),\theta(v),\theta(v)))\\ &= \Psi(\alpha(0,\theta(v),\theta(v),\theta(v)))\\ &< \Theta(\alpha(0,\theta(v),\theta(v),\theta(v))),\end{aligned}$$

a contradiction, so $\Lambda(\Im) \subseteq Q_\theta$.

Next, we prove that $\lim_{k\to\infty} G(v_k, v_{k+1}, v_{k+1}) = 0$, $\lim_{k\to\infty} \theta(v_k) = 0$ and $\lim_{k\to\infty} \theta(v_{k+1}) = 0$. Assume that $v_0 \in \Omega$ and $\{v_k\}$ is a sequence defined as $v_k = \Im v_{k-1}$ for all $k \in \mathbb{N}$. If there is $k_0 \in \mathbb{N}$ so that $v_{k_0} = v_{k_0+1}$, that is $v_{k_0} = \Im v_{k_0}$. Hence, v_{k_0} is a FP of $\Im$. Clearly, in this case, $\lim_{k\to\infty} G(v_k, v_{k+1}, v_{k+1}) = 0$ and $\lim_{k\to\infty} \theta(v_k) = 0$ and the proof is finished. So, we assume that $G(v_k, v_{k+1}, v_{k+1}) > 0$ and set $v = v_k$, $\varrho = v_{k+1}$ and $\sigma = v_{k+1}$ in (1); then, we obtain

$$\begin{aligned}&\Theta(\alpha(G(v_{k+1},v_{k+2},v_{k+2}),\theta(v_{k+1}),\theta(v_{k+2}),\theta(v_{k+2})))\\ =\;&\Theta(\alpha(G(\Im v_k,\Im v_{k+1},\Im v_{k+1}),\theta(\Im v_k),\theta(\Im v_{k+1}),\theta(\Im v_{k+1})))\\ \leq\;&\Psi(\alpha(G(v_k,v_{k+1},v_{k+1}),\theta(v_k),\theta(v_{k+1}),\theta(v_{k+1})))\\ <\;&\Theta(\alpha(G(v_k,v_{k+1},v_{k+1}),\theta(v_k),\theta(v_{k+1}),\theta(v_{k+1}))).\end{aligned} \tag{2}$$

Since Θ is nondecreasing, we find that

$$\alpha(G(v_{k+1},v_{k+2},v_{k+2}),\theta(v_{k+1}),\theta(v_{k+2}),\theta(v_{k+2})) < \alpha(G(v_k,v_{k+1},v_{k+1}),\theta(v_k),\theta(v_{k+1}),\theta(v_{k+1})).$$

Hence, the sequence $\{\alpha(G(v_k,v_{k+1},v_{k+1}),\theta(v_k),\theta(v_{k+1}),\theta(v_{k+1}))\}$ is decreasing with a lower bound. So, there is $\epsilon \geq 0$ so that $\lim_{k\to\infty} \alpha_k = \epsilon$, where

$$\alpha_k = \alpha(G(v_k,v_{k+1},v_{k+1}),\theta(v_k),\theta(v_{k+1}),\theta(v_{k+1})).$$

If $\epsilon > 0$, then taking lim sup on both sides of (2), by (T_2) and (T_3), one has

$$\begin{aligned}\Theta(\epsilon) &= \limsup_{\alpha_{k+1}\to\epsilon} \Theta(\alpha_{k+1}) \leq \limsup_{\alpha_k\to\epsilon} \Psi(\alpha_k)\\ &\leq \limsup_{\tau\to\epsilon} \Psi(\tau) < \Theta(\epsilon),\end{aligned}$$

this is a contradiction. So, $\epsilon = 0$. It follows by (α_1) that

$$\max\{G(v_{k+1},v_{k+2},v_{k+2}),\theta(v_{k+1}),\theta(v_{k+2})\} \leq \alpha(G(v_{k+1},v_{k+2},v_{k+2}),\theta(v_{k+1}),\theta(v_{k+2}),\theta(v_{k+2}))$$

Obviously,

$$G(v_{k+1},v_{k+2},v_{k+2}) \leq \alpha(G(v_{k+1},v_{k+2},v_{k+2}),\theta(v_{k+1}),\theta(v_{k+2}),\theta(v_{k+2})),$$

$$\theta(v_{k+1}) \leq \alpha(G(v_{k+1},v_{k+2},v_{k+2}),\theta(v_{k+1}),\theta(v_{k+2}),\theta(v_{k+2})),$$

and

$$\theta(v_{k+2}) \leq \alpha(G(v_{k+1},v_{k+2},v_{k+2}),\theta(v_{k+1}),\theta(v_{k+2}),\theta(v_{k+2})).$$

Passing $k \to \infty$ in the three above inequalities, one can write

$$\begin{aligned}0 &\leq \lim_{k\to\infty} G(v_{k+1},v_{k+2},v_{k+2}) \leq \lim_{k\to\infty} \alpha(G(v_{k+1},v_{k+2},v_{k+2}),\theta(v_{k+1}),\theta(v_{k+2}),\theta(v_{k+2})) = 0,\\ 0 &\leq \lim_{k\to\infty} \theta(v_{k+1}) \leq \lim_{k\to\infty} \alpha(G(v_{k+1},v_{k+2},v_{k+2}),\theta(v_{k+1}),\theta(v_{k+2}),\theta(v_{k+2})) = 0,\\ 0 &\leq \lim_{k\to\infty} \theta(v_{k+2}) \leq \lim_{k\to\infty} \alpha(G(v_{k+1},v_{k+2},v_{k+2}),\theta(v_{k+1}),\theta(v_{k+2}),\theta(v_{k+2})) = 0,\end{aligned}$$

that is $\lim_{k\to\infty} G(v_{k+1}, v_{k+2}, v_{k+2}) = 0$, $\lim_{k\to\infty} \theta(v_{k+1}) = 0$ and $\lim_{k\to\infty} \theta(v_{k+2}) = 0$. By induction, we obtain that

$$\lim_{k\to\infty} G(v_k, v_{k+1}, v_{k+1}) = 0, \ \lim_{k\to\infty} \theta(v_k) = 0 \text{ and } \lim_{k\to\infty} \theta(v_{k+1}) = 0. \tag{3}$$

After that, we show that $\{v_k\}$ is a G-Cauchy sequence. Assuming the opposite, then there is $\epsilon > 0$ and two sequences $\{v_{k(j)}\}$ and $\{v_{l(j)}\}$, where $k(j)$ and $l(j)$ are two positive integers with $k(j) > l(j)$ so that

$$G\Big(v_{l(j)}, v_{k(j)}, v_{k(j)}\Big) \geq \epsilon, \ G\Big(v_{l(j)}, v_{k(j)-1}, v_{k(j)-1}\Big) < \epsilon \text{ and } G\Big(v_{l(j)-1}, v_{k(j)}, v_{k(j)}\Big) < \epsilon.$$

Using the rectangle inequality, we have

$$\begin{aligned} \epsilon &\leq G\Big(v_{l(j)}, v_{k(j)}, v_{k(j)}\Big) \\ &\leq G\Big(v_{l(j)}, v_{k(j)-1}, v_{k(j)-1}\Big) + G\Big(v_{k(j)-1}, v_{k(j)}, v_{k(j)}\Big) \\ &< \epsilon + G\Big(v_{k(j)-1}, v_{k(j)}, v_{k(j)}\Big). \end{aligned} \tag{4}$$

As $j \to \infty$ in (4), we conclude that

$$\lim_{j\to\infty} G\Big(v_{l(j)}, v_{k(j)}, v_{k(j)}\Big) = \epsilon. \tag{5}$$

Again, applying the rectangle inequality, one can write

$$G\Big(v_{l(j)-1}, v_{k(j)-1}, v_{k(j)-1}\Big) \leq G\Big(v_{l(j)-1}, v_{k(j)}, v_{k(j)}\Big) + G\Big(v_{k(j)}, v_{k(j)-1}, v_{k(j)-1}\Big), \tag{6}$$

and

$$G\Big(v_{l(j)}, v_{k(j)}, v_{k(j)}\Big) \leq G\Big(v_{l(j)}, v_{l(j)-1}, v_{l(j)-1}\Big) + G\Big(v_{l(j)-1}, v_{k(j)-1}, v_{k(j)-1}\Big) + G\Big(v_{k(j)-1}, v_{k(j)}, v_{k(j)}\Big). \tag{7}$$

Taking $j \to \infty$ in (6) and (7), one has

$$\lim_{j\to\infty} G\Big(v_{l(j)-1}, v_{k(j)-1}, v_{k(j)-1}\Big) = \epsilon. \tag{8}$$

Set $v = v_{l(j)-1}$, $\varrho = v_{k(j)-1}$ and $\sigma = v_{k(j)-1}$ in (1) and using (T_1), we obtain

$$\begin{aligned} \Theta(\alpha_{l,k}) &= \Theta\Big(\alpha\Big(G\Big(\Im v_{l(j)-1}, \Im v_{k(j)-1}, \Im v_{k(j)-1}\Big), \theta\Big(\Im v_{l(j)-1}\Big), \theta\Big(\Im v_{k(j)-1}\Big), \theta\Big(\Im v_{k(j)-1}\Big)\Big)\Big) \\ &\leq \Psi(\alpha_{l-1,k-1}), \end{aligned} \tag{9}$$

where $\alpha_{l,k} = \alpha\Big(G\Big(\Im v_{l(j)}, \Im v_{k(j)}, \Im v_{k(j)}\Big), \theta\Big(\Im v_{l(j)}\Big), \theta\Big(\Im v_{k(j)}\Big), \theta\Big(\Im v_{k(j)}\Big)\Big)$. Passing lim sup in (9), we have

$$\begin{aligned} \Theta(\epsilon) &= \limsup_{\alpha_{l,k}\to\epsilon^+} \Theta(\alpha_{l,k}) \leq \limsup_{\alpha_{l-1,k-1}\to\epsilon^+} \Psi(\alpha_{l-1,k-1}) \\ &\leq \limsup_{\tau\to\epsilon^+} \Psi(\tau) < \Theta(\epsilon), \end{aligned}$$

which leads to a contradiction. Hence, $\{v_k\}$ is a G-Cauchy sequence on Ω. Since Ω is G-complete, then there is $v^* \in \Omega$ so that $\{v_k\}$ is G-convergent to v^*. Because θ is lsc, we obtain that

$$\theta(v^*) \leq \liminf_{k\to\infty} \theta(v_{k+1}) \leq \lim_{k\to\infty} \theta(v_{k+1}) = 0.$$

Hence, $\theta(v^*) = 0$. Now, we prove that $v^* = \Im v^*$. If there is $k_0 \in \mathbb{N}$ so that for $k \geq k_0$, $v_k = \Im v^*$ is still true. Then, $\Im v^* = \lim_{k\to\infty} v_k = v^*$, that is $v^* = \Im v^*$. Suppose that $G(v^*, \Im v^*, \Im v^*) > 0$. Put $v = v_k$, $\varrho = v^*$ and $\sigma = v^*$ in (1); then, we have

$$\Theta(\alpha(G(\Im v_k, \Im v^*, \Im v^*), \theta(\Im v_k), \theta(\Im v^*), \theta(\Im v^*))) \leq \Psi(\alpha(G(v_k, v^*, v^*), \theta(v_k), 0, 0)).$$

From (T_1) and (T_2), we obtain

$$\begin{aligned} G(v_{k+1}, \Im v^*, \Im v^*) &\leq \alpha(G(\Im v_k, \Im v^*, \Im v^*), \theta(\Im v_k), \theta(\Im v^*), \theta(\Im v^*)) \\ &< \alpha(G(v_k, v^*, v^*), \theta(v_k), 0, 0). \end{aligned} \tag{10}$$

Letting $k \to \infty$ in (10), we obtain that

$$G(v^*, \Im v^*, \Im v^*) \leq 0,$$

which is a contradiction. Hence, $G(v^*, \Im v^*, \Im v^*) = 0$, that is $v^* = \Im v^*$.

In the last step, we claim that $v^* = \varrho$, for $v^*, \varrho \in \Lambda(\Im)$.

Conversely, assume that $v^* \neq \varrho$, then putting $v = v^*$ and $\sigma = \varrho$ in (1), one has

$$\begin{aligned} \Theta(\alpha(G(v^*, \varrho, \varrho), 0, 0, 0)) &= \Theta(\alpha(G(v^*, \varrho, \varrho), \theta(v^*), \theta(\varrho), \theta(v^*))) \\ &= \Theta(\alpha(G(\Im v^*, \Im\varrho, \Im\varrho), \theta(\Im v^*), \theta(\Im\varrho), \theta(\Im\varrho))) \\ &\leq \Psi(\alpha(G(v^*, \varrho, \varrho), \theta(v^*), \theta(\varrho), \theta(\varrho))) \\ &= \Psi(\alpha(G(v^*, \varrho, \varrho), 0, 0, 0)) \\ &< \Theta(\alpha(G(v^*, \varrho, \varrho), 0, 0, 0)), \end{aligned}$$

which is a contradiction. So, $v^* = \varrho$. Hence, $\Im$ is a θ-Picard mapping. □

Remark 3.

(i) *Put $\alpha = \alpha_1$, $\Theta(\tau) = \tau$, for $\tau > 0$ and $\theta(v) = 0$. Then, Theorem 3 reduces to (Theorem 2.1, [31]).*
(ii) *Set $\alpha = \alpha_1$ and $\Psi(\tau) = \Theta(\tau) - \Psi(\tau)$ for $\tau > 0$ and $\theta(v) = 0$ in Theorem 3; then, we have (Theorem 2.6, [31]).*
(iii) *If we take $\theta(v) = 0$, $\alpha = \alpha_1$, $\Theta(\tau) = \tau$, $\Psi(\tau) = \delta\tau$, for $\tau > 0$, where $\delta \in [0, 1)$ in Theorem 3, we obtain (Theorem 2.1, [32]).*
(iv) *Take $\theta(v) = 0$, $\alpha = \alpha_1$, $\Theta(\tau) = \tau$, $\Psi(\tau) = \tau - \Theta(\tau)$, for $\tau > 0$, in Theorem 3, we obtain (Theorem 3.2, [9]).*

Now, in order to support Theorem 3, we give an example below:

Example 4. *Let $\Omega = [0, \infty)$ and $G : \Omega^3 \to \mathbb{R}$ be defined by*

$$G(v, \varrho, \sigma) = \begin{cases} 0, & \text{if } v = \varrho = \sigma, \\ \max\{v, \varrho, \sigma\}, & \text{otherwise.} \end{cases}$$

Clearly, (Ω, G) is a complete GMS. Assume that $\Im : \Omega \to \Omega$ is a mapping defined as

$$\Im v = \begin{cases} \frac{1}{4}v, & \text{if } v \in [0, 2), \\ \frac{1}{9}v, & \text{otherwise.} \end{cases}$$

Describe the functions $\Theta, \alpha, \theta, \Psi$ as $\Theta(\tau) = \tau$, $\alpha(v, \varrho, \sigma, \ell) = v + \varrho + \sigma + \ell$, $\theta(v) = v$ and $\Psi(\tau) = \frac{1}{3}\tau$, for $\tau \in \mathbb{R}^+$, respectively.

Now, to verify the condition (1) of Theorem 3, we discuss the following cases:

(Ca. I) *If* $v = \varrho = \sigma \in [0,2)$. *Then*

$$\begin{aligned}
&\Theta(\alpha(G(\Im v, \Im\varrho, \Im\sigma), \theta(\Im v), \theta(\Im\varrho), \theta(\Im\sigma))) \\
= \;& G(\Im v, \Im\varrho, \Im\sigma) + \theta(\Im v) + \theta(\Im\varrho) + \theta(\Im\sigma) \\
= \;& 0 + \Im v + \Im\varrho + \Im\sigma \\
= \;& \frac{1}{4}(v + \varrho + \sigma) \\
\leq \;& \frac{1}{3}(0 + \theta(v) + \theta(v) + \theta(v)) \\
= \;& \frac{1}{3}(0 + \theta(v) + \theta(v) + \theta(v)) \\
\leq \;& \Psi(\alpha(G(v, \varrho, \sigma), \theta(v), \theta(\varrho), \theta(\sigma))).
\end{aligned}$$

(Ca.II) *If* $v \in [0,2)$ *and* $\varrho = \sigma \in \Omega - [0,2)$. *Then*

$$\begin{aligned}
&\Theta(\alpha(G(\Im v, \Im\varrho, \Im\sigma), \theta(\Im v), \theta(\Im\varrho), \theta(\Im\sigma))) \\
= \;& G(\Im v, \Im\varrho, \Im\sigma) + \theta(\Im v) + \theta(\Im\varrho) + \theta(\Im\sigma) \\
= \;& \max\{\Im v, \Im\varrho, \Im\sigma\} + \Im v + \Im\varrho + \Im\sigma \\
= \;& \max\{\frac{v}{4}, \frac{\varrho}{9}, \frac{\sigma}{9}\} + \frac{v}{4} + \frac{\varrho}{9} + \frac{\sigma}{9} \\
\leq \;& \max\{\frac{v}{4}, \frac{\varrho}{4}, \frac{\sigma}{4}\} + \frac{v}{4} + \frac{\varrho}{4} + \frac{\sigma}{4} \\
= \;& \frac{1}{4}\max\{v, \varrho, \sigma\} + v + \varrho + \sigma \\
\leq \;& \frac{1}{3}(G(v, \varrho, \sigma) + \theta(v) + \theta(v) + \theta(v)) \\
= \;& \Psi(\alpha(G(v, \varrho, \sigma), \theta(v), \theta(\varrho), \theta(\sigma))).
\end{aligned}$$

(Ca.III) *If* $\sigma = \varrho \in [0,2)$ *and* $v \in \Omega - [0,2)$. *Then*

$$\begin{aligned}
&\Theta(\alpha(G(\Im v, \Im\varrho, \Im\sigma), \theta(\Im v), \theta(\Im\varrho), \theta(\Im\sigma))) \\
= \;& G(\Im v, \Im\varrho, \Im\sigma) + \theta(\Im v) + \theta(\Im\varrho) + \theta(\Im\sigma) \\
= \;& \max\{\Im v, \Im\varrho, \Im\sigma\} + \Im v + \Im\varrho + \Im\sigma \\
= \;& \max\{\frac{v}{9}, \frac{\varrho}{4}, \frac{\sigma}{4}\} + \frac{v}{9} + \frac{\varrho}{4} + \frac{\sigma}{4} \\
\leq \;& \max\{\frac{v}{4}, \frac{\varrho}{4}, \frac{\sigma}{4}\} + \frac{v}{4} + \frac{\varrho}{4} + \frac{\sigma}{4} \\
= \;& \frac{1}{4}\max\{v, \varrho, \sigma\} + v + \varrho + \sigma \\
\leq \;& \frac{1}{3}(G(v, \varrho, \sigma) + \theta(v) + \theta(v) + \theta(v)) \\
= \;& \Psi(\alpha(G(v, \varrho, \sigma), \theta(v), \theta(\varrho), \theta(\sigma))).
\end{aligned}$$

(Ca.IV) *If* $v = \varrho = \sigma \in \Omega - [0, 2)$. *Then*

$$\begin{aligned}
& \Theta(\alpha(G(\Im v, \Im \varrho, \Im \sigma), \theta(\Im v), \theta(\Im \varrho), \theta(\Im \sigma))) \\
= \; & G(\Im v, \Im \varrho, \Im \sigma) + \theta(\Im v) + \theta(\Im \varrho) + \theta(\Im \sigma) \\
= \; & 0 + \Im v + \Im \varrho + \Im \sigma \\
= \; & \frac{v}{9} + \frac{\varrho}{9} + \frac{\sigma}{9} \\
\leq \; & \frac{1}{4}(v + \varrho + \sigma) \\
= \; & \frac{1}{4}(0 + v + \varrho + \sigma) \\
\leq \; & \frac{1}{3}(G(v, \varrho, \sigma) + \theta(v) + \theta(v) + \theta(v)) \\
= \; & \Psi(\alpha(G(v, \varrho, \sigma), \theta(v), \theta(\varrho), \theta(\sigma))).
\end{aligned}$$

Based on the above cases, we conclude that $\Im$ *is* $(\Theta, \alpha, \theta, \Psi)$ *contraction mapping. Therefore, all requirements of Theorem 3 are fulfilled and* $\Im$ *has a unique FP* $(0, 0, 0)$.

Corollary 1. *Let* (Ω, G) *be a complete GMS and* $\Im : \Omega \to \Omega$ *be a self-mapping. Assume that there exists* $\delta \in [0, 1)$ *so that* $\alpha(G(\Im v, \Im \varrho, \Im \sigma), \theta(\Im v), \theta(\Im \varrho), \theta(\Im \sigma)) > 0$, *implies*

$$\Theta(\alpha(G(\Im v, \Im \varrho, \Im \sigma), \theta(\Im v), \theta(\Im \varrho), \theta(\Im \sigma))) \leq \Theta(\alpha(G(v, \varrho, \sigma), \theta(v), \theta(\varrho), \theta(\sigma)))^{\delta},$$

for all $v, \varrho, \sigma \in \Omega$, *where* θ *is lsc and* $\Theta : \mathbb{R}^+ \to \mathbb{R}$ *is nondecreasing. Then,* $\Im$ *is a* θ*-Picard mapping.*

Proof. The result follows immediately by putting $\Psi(\tau) = \Theta(\tau)^{\delta}$, for $\tau > 0$ in Theorem 3. □

Corollary 2. *Let* (Ω, G) *be a complete GMS and* $\Im : \Omega \to \Omega$ *be a self-mapping. Suppose that there exists* $p > 0$ *so that* $\alpha(G(\Im v, \Im \varrho, \Im \sigma), \theta(\Im v), \theta(\Im \varrho), \theta(\Im \sigma)) > 0$, *implies*

$$\Theta(\alpha(G(\Im v, \Im \varrho, \Im \sigma), \theta(\Im v), \theta(\Im \varrho), \theta(\Im \sigma))) \leq \Theta(\alpha(G(v, \varrho, \sigma), \theta(v), \theta(\varrho), \theta(\sigma))) - p,$$

for all $v, \varrho, \sigma \in \Omega$, *where* θ *is lsc and* $\Theta : \mathbb{R}^+ \to \mathbb{R}$ *is nondecreasing. Then,* $\Im$ *is a* θ*-Picard mapping.*

Proof. Setting $\Psi(\tau) = \Theta(\tau) - p$. The result follows by Theorem 3. □

Corollary 3. *Let* (Ω, G) *be a complete GMS and* $\Im$: $\Omega arrow \Omega$ *be a self-mapping. Suppose that there exists* $(p, q) \in (\Theta, \Psi)$ *with continuous* q, *so that for each* $v, \varrho, \sigma \in \Omega$ *with* $\alpha(G(\Im v, \Im \varrho, \Im \sigma), \theta(\Im v), \theta(\Im \varrho), \theta(\Im \sigma)) > 0$,

$$p(\alpha(G(\Im v, \Im \varrho, \Im \sigma), \theta(\Im v), \theta(\Im \varrho), \theta(\Im \sigma))) \leq q(p(\alpha(G(v, \varrho, \sigma), \theta(v), \theta(\varrho), \theta(\sigma)))) - p,$$

and θ *is lsc. Then,* $\Im$ *is a* θ*-Picard mapping.*

Proof. The proof is proven by Theorem 3 and Remark 2. □

Corollary 4. *Let* (Ω, G) *be a complete GMS and* $\Im : \Omega \to \Omega$ *be a self-mapping. Assume that* $\xi : \mathbb{R}^+ \to \mathbb{R}$ *and* $q : \mathbb{R}^+ \to \mathbb{R}^+$ *are functions so that*

(a) *q is a nondecreasing;*
(b) $\limsup_{\tau \to \epsilon} \xi(\tau) < \xi(\epsilon)$, *for all* $\tau > 0$;
(c) *For all* $\epsilon > 0$, $\liminf_{\tau \to \epsilon^+} q(\tau) > 0$.

Suppose also, for each $v, \varrho, \sigma \in \Omega$ with $\alpha(G(\Im v, \Im\varrho, \Im\sigma), \theta(\Im v), \theta(\Im\varrho), \theta(\Im\sigma)) > 0$, $\Im$ fulfills

$$\begin{aligned}\xi(\alpha(G(\Im v, \Im\varrho, \Im\sigma), \theta(\Im v), \theta(\Im\varrho), \theta(\Im\sigma))) \leq\ & \xi(\alpha(G(v, \varrho, \sigma), \theta(v), \theta(\varrho), \theta(\sigma))) \\ & -q(\alpha(G(v, \varrho, \sigma), \theta(v), \theta(\varrho), \theta(\sigma))),\end{aligned}$$

and θ is lsc. Then, $\Im$ is a θ-Picard mapping.

Proof. It is given from Theorem 3 by putting $\Theta(\tau) = \xi(\tau)$ and $\Psi(\tau) = \xi(\tau) - q(\tau)$, for $\tau > 0$. □

Corollary 5. *Let (Ω, G) be a complete GMS and $\Im : \Omega \to \Omega$ be a given-mapping. Suppose that there exists a continuous function $q \in \Psi$ so that for each $v, \varrho, \sigma \in \Omega$, $\Im$ fulfills*

$$\alpha(G(\Im v, \Im\varrho, \Im\sigma), \theta(\Im v), \theta(\Im\varrho), \theta(\Im\sigma)) \leq q(\alpha(G(v, \varrho, \sigma), \theta(v), \theta(\varrho), \theta(\sigma))),$$

and θ is lsc. Then, $\Im$ is a θ-Picard mapping.

Proof. The result follows immediately by Corollary 3. □

Corollary 6. *Let (Ω, G) be a complete GMS and $\Im : \Omega \to \Omega$ be a self-mapping. Assume that there are continuous functions $p \in \Theta$ and $q \in \Psi$ so that for each $v, \varrho, \sigma \in \Omega$ with $\alpha(G(\Im v, \Im\varrho, \Im\sigma), \theta(\Im v), \theta(\Im\varrho), \theta(\Im\sigma)) > 0$,*

$$p(G(\Im v, \Im\varrho, \Im\sigma)) \leq q(p(G(v, \varrho, \sigma))),$$

Then, $\Im$ has a unique FP.

Proof. The proof is obtained from Corollary 3 by taking $\alpha = \alpha_1$ and $\theta(v) = 0$ for $v \in \Omega$. □

Now, another sort of contraction, known as a rational $(\Theta, \alpha, \theta, \Psi)$ contraction, can be stated as follows:

Definition 10. *Let (Ω, G) be a GMS. The mapping $\Im : \Omega \to \Omega$ is called a rational $(\Theta, \alpha, \theta, \Psi)$-contraction if there are Θ, Ψ that fulfill the conditions $(T_1) - (T_3)$ in Theorem 1 so that for all $v, \varrho, \sigma \in \Omega$ with $\alpha(G(\Im v, \Im\varrho, \Im\sigma), \theta(\Im v), \theta(\Im\varrho), \theta(\Im\sigma)) > 0$, $\Im$ justifies*

$$\Theta(\alpha(G(\Im v, \Im\varrho, \Im\sigma), \theta(\Im v), \theta(\Im\varrho), \theta(\Im\sigma))) \leq \Psi(\alpha(U(v, \varrho, \sigma), \theta(v), \theta(\varrho), \theta(\sigma))), \tag{11}$$

where

$$U(v, \varrho, \sigma) = \max\left\{G(v, \varrho, \sigma), \frac{G(v, \Im v, \Im v)(1 + G(\varrho, \Im\varrho, \Im\varrho) + G(\sigma, \Im\sigma, \Im\sigma))}{1 + 2G(\Im v, \Im\varrho, \Im\sigma)}\right\}.$$

Theorem 4. *Let (Ω, G) be a complete GMS and $\Im : \Omega \to \Omega$ be a rational $(\Theta, \alpha, \theta, \Psi)$ contraction mapping. Then, $\Im$ is a θ-Picard mapping provided that θ is lsc.*

Proof. Firstly, we illustrate that $\Lambda(\Im) \subseteq Q_\theta$. Let there be $v \in \Lambda(\Im)$ so that $\theta(v) \neq 0$. Putting $v = \varrho = \sigma$ in (11), we have

$$\begin{aligned}\Theta(\alpha(0, \theta(v), \theta(v), \theta(v))) &= \Theta(\alpha(G(\Im v, \Im v, \Im v), \theta(\Im v), \theta(\Im v), \theta(\Im v))) \\ &\leq \Psi(\alpha(U(v, v, v), \theta(v), \theta(v), \theta(v))) \\ &= \Psi(\alpha(0, \theta(v), \theta(v), \theta(v))) \\ &< \Theta(\alpha(0, \theta(v), \theta(v), \theta(v))),\end{aligned}$$

which is a contradiction, so $\Lambda(\Im) \subseteq Q_\theta$.

After that, we claim that $\lim_{k\to\infty} G(v_k, v_{k+1}, v_{k+1}) = 0$, $\lim_{k\to\infty} \theta(v_k) = 0$ and $\lim_{k\to\infty} \theta(v_{k+1}) = 0$. Let $v_0 \in \Omega$ and $\{v_k\}$ be a sequence described as $v_k = \Im v_{k-1}$ for all $k \in \mathbb{N}$. If there is $k_0 \in \mathbb{N}$ so that $v_{k_0} = v_{k_0+1}$, that is $v_{k_0} = \Im v_{k_0}$. Hence, v_{k_0} is a FP of $\Im$. Clearly, in this case, $\lim_{k\to\infty} G(v_k, v_{k+1}, v_{k+1}) = 0$, $\lim_{k\to\infty} \theta(v_k) = 0$ and $\lim_{k\to\infty} \theta(v_{k+1}) = 0$. Hence, the proof is completed. So, we assume that $G(v_k, v_{k+1}, v_{k+1}) > 0$. Setting $v = v_k$, $\varrho = v_{k+1}$ and $\sigma = v_{k+1}$ in (11), we obtain that

$$\begin{aligned} & \Theta(\alpha(G(v_{k+1}, v_{k+2}, v_{k+2}), \theta(v_{k+1}), \theta(v_{k+2}), \theta(v_{k+2}))) \\ = \; & \Theta(\alpha(G(\Im v_k, \Im v_{k+1}, \Im v_{k+1}), \theta(\Im v_k), \theta(\Im v_{k+1}), \theta(\Im v_{k+1}))) \\ \leq \; & \Psi(\alpha(U(v_k, v_{k+1}, v_{k+1}), \theta(v_k), \theta(v_{k+1}), \theta(v_{k+1}))), \end{aligned}$$

where

$$\begin{aligned} U(v_k, v_{k+1}, v_{k+1}) &= \max\left\{ \begin{array}{c} G(v_k, v_{k+1}, v_{k+1}), \\ \frac{G(v_k, \Im v_k, \Im v_k)(1+G(v_{k+1}, \Im v_{k+1}, \Im v_{k+1})+G(v_{k+1}, \Im v_{k+1}, \Im v_{k+1}))}{1+2G(\Im v_k, \Im v_{k+1}, \Im v_{k+1})} \end{array} \right\} \\ &= \max\left\{ \begin{array}{c} G(v_k, v_{k+1}, v_{k+1}), \\ \frac{G(v_k, v_{k+1}, v_{k+1})(1+G(v_{k+1}, v_{k+2}, v_{k+2})+G(v_{k+1}, v_{k+2}, v_{k+2}))}{1+2G(v_{k+1}, v_{k+2}, v_{k+2})} \end{array} \right\} \\ &= G(v_k, v_{k+1}, v_{k+1}). \end{aligned}$$

Hence, we obtain (3) in a manner similar to the proof of Theorem 3. Next, we claim that $\{v_k\}$ is a G-Cauchy sequence. If it is not, then by the same method of the proof of Theorem 3, we deduce that (5) and (8). Taking $v = v_{l(j)-1}$, $\varrho = v_{k(j)-1}$ and $\sigma = v_{k(j)-1}$ in (11) and using (T_1), we have

$$\begin{aligned} & \Theta\Big(\alpha\Big(G\Big(v_{l(j)}, v_{k(j)}, v_{k(j)}\Big), \theta\Big(v_{l(j)}\Big), \theta\Big(v_{k(j)}\Big), \theta\Big(v_{k(j)}\Big)\Big)\Big) \\ = \; & \Theta\Big(\alpha\Big(G\Big(\Im v_{l(j)-1}, \Im v_{k(j)-1}, \Im v_{k(j)-1}\Big), \theta\Big(\Im v_{l(j)-1}\Big), \theta\Big(\Im v_{k(j)-1}\Big), \theta\Big(\Im v_{k(j)-1}\Big)\Big)\Big) \\ \leq \; & \Psi(\alpha\Big(U\Big(v_{l(j)-1}, v_{k(j)-1}, v_{k(j)-1}\Big), \theta\Big(v_{l(j)-1}\Big), \theta\Big(v_{k(j)-1}\Big), \theta\Big(v_{k(j)-1}\Big)\Big)), \end{aligned}$$

where

$$\begin{aligned} & U\Big(v_{l(j)-1}, v_{k(j)-1}, v_{k(j)-1}\Big) \\ = \; & \max\left\{ \begin{array}{c} G\Big(v_{l(j)-1}, v_{k(j)-1}, v_{k(j)-1}\Big), \\ \frac{G\big(v_{l(j)-1}, \Im v_{l(j)-1}, \Im v_{l(j)-1}\big)\big(1+G\big(v_{k(j)-1}, \Im v_{k(j)-1}, \Im v_{k(j)-1}\big)+G\big(v_{k(j)-1}, \Im v_{k(j)-1}, \Im v_{k(j)-1}\big)\big)}{1+2G\big(\Im v_{l(j)-1}, \Im v_{k(j)-1}, \Im v_{k(j)-1}\big)} \end{array} \right\} \\ = \; & \max\left\{ \begin{array}{c} G\Big(v_{l(j)-1}, v_{k(j)-1}, v_{k(j)-1}\Big), \\ \frac{G\big(v_{l(j)-1}, \Im v_{l(j)-1}, \Im v_{l(j)-1}\big)\big(1+G\big(v_{k(j)-1}, v_{k(j)}, v_{k(j)}\big)+G\big(v_{k(j)-1}, v_{k(j)}, v_{k(j)}\big)\big)}{1+2G\big(v_{l(j)}, v_{k(j)}, v_{k(j)}\big)} \end{array} \right\}. \end{aligned}$$

It follows from (3) and (8) that

$$\lim_{j\to\infty} U\Big(v_{l(j)-1}, v_{k(j)-1}, v_{k(j)-1}\Big) = \varepsilon.$$

Following the same steps as the proof of Theorem 3, we conclude that $\{v_k\}$ is a G-Cauchy sequence in a complete GMS (Ω, G), and there is $v^* \in \Omega$ so that $\{v_k\}$ is G-convergent to v^*, that is $v_k \to v^*$ as $k \to \infty$ and $\theta(v^*) = 0$.

Now, we show that $v^* = \Im v^*$. If there is $k_0 \in \mathbb{N}$ so that for $k \geq k_0$, $v_k = \Im v^*$ is still true, then $\Im v^* = \lim_{k\to\infty} v_k = v^*$, that is $v^* = \Im v^*$. On the contrary, put $G(v^*, \Im v^*, \Im v^*) > 0$. Putting $v = v_k$, $\varrho = v^*$ and $\sigma = v^*$ in (11), we obtain

$$\Theta(\alpha(G(\Im v_k, \Im v^*, \Im v^*), \theta(\Im v_k), \theta(\Im v^*), \theta(\Im v^*))) \leq \Psi(\alpha(U(v_k, v^*, v^*), \theta(v_k), 0, 0)).$$

It follows from (T_1) and (T_2) that

$$\begin{aligned} G(v_{k+1},\Im v^*,\Im v^*) &\leq \alpha(G(\Im v_k,\Im v^*,\Im v^*),\theta(\Im v_k),\theta(\Im v^*),\theta(\Im v^*)) \\ &< \alpha(U(v_k,v^*,v^*),\theta(v_k),0,0), \end{aligned} \tag{12}$$

where

$$U(v_k,v^*,v^*) = \max\left\{G(v_k,v^*,v^*), \frac{G(v_k,\Im v_k,\Im v_k)(1+2G(v^*,\Im v^*,\Im v^*))}{1+2G(\Im v,\Im v^*,\Im v^*)}\right\}$$

Taking $k \to \infty$ in (12), we have

$$G(v^*,\Im v^*,\Im v^*) \leq 0,$$

that is, $v^* = \Im v^*$.

Ultimately, we prove that $v^* = \varrho$, for $v^*,\varrho \in \Lambda(\Im)$. If $v^* \neq \varrho$, then setting $v = v^*$ and $\sigma = \varrho$ in (11), one has

$$\begin{aligned} & \Theta(\alpha(G(v^*,\varrho,\varrho),0,0,0)) \\ = \; & \Theta(\alpha(G(\Im v^*,\Im\varrho,\Im\varrho),\theta(\Im v^*),\theta(\Im\varrho),\theta(\Im\varrho))) \\ \leq \; & \Psi(\alpha(N(v^*,\varrho,\varrho),\theta(v^*),\theta(\varrho),\theta(\varrho))) \\ = \; & \Psi\left(\alpha\left(\max\left\{G(v^*,\varrho,\varrho), \frac{G(v^*,\Im v^*,\Im v^*)(1+2G(\varrho,\Im\varrho,\Im\varrho))}{1+2G(\Im v^*,\Im\varrho,\Im\varrho)}\right\},0,0,0\right)\right) \\ = \; & \Psi(\alpha(G(v^*,\varrho,\varrho),0,0,0)) \\ < \; & \Theta(\alpha(G(v^*,\varrho,\varrho),0,0,0)), \end{aligned}$$

which is a contradiction. So, $v^* = \varrho$. Hence, $\Im$ is a θ-Picard mapping. □

The following example supports Theorem 4:

Example 5. *Let $\Omega = [0,1]$ and G be the G-Metric on Ω^3 defined by $G(v,\varrho,\sigma) = |v-\varrho| + |\varrho-\sigma| + |v-\sigma|$, for all $v,\varrho,\sigma \in \Omega$. Obviously, (Ω, G) is a complete GMS. Define the mapping $\Im : \Omega \to \Omega$ by*

$$\Im v = \begin{cases} \frac{v}{6}, & \text{if } v \in [0,\frac{1}{6}), \\ \frac{v}{12}, & \text{if } v \in [\frac{1}{6},1]. \end{cases}$$

Let the functions $\Theta,\alpha,\theta,\Psi$ be defined by $\Theta(\tau) = \tau$, $\alpha(v,\varrho,\sigma,\ell) = \max\{v,\varrho,\sigma,\ell\}$, $\theta(v) = v$ and $\Psi(\tau) = \frac{1}{6}\tau$, for $\tau \in \mathbb{R}^+$, respectively.

To satisfy the contractive condition (11), we discuss the following cases:

(i) If $v = \varrho = \sigma \in [0,\frac{1}{6})$. Then,

$$\begin{aligned} & \Theta(\alpha(G(\Im v,\Im\varrho,\Im\sigma),\theta(\Im v),\theta(\Im\varrho),\theta(\Im\sigma))) \\ = \; & \max\{G(\Im v,\Im\varrho,\Im\sigma),\theta(\Im v),\theta(\Im v),\theta(\Im v)\} \\ = \; & \max\left\{0,\frac{v}{6},\frac{v}{6},\frac{v}{6}\right\} = \frac{v}{6} \leq \frac{1}{6}\left(\frac{5}{3}v + \frac{50}{9}v^2\right) \\ = \; & \frac{1}{6}\max\left\{0, \frac{\frac{5v}{3}(1+\frac{5v}{3}+\frac{5v}{3})}{1+0}\right\} \\ = \; & \frac{1}{6}\max\left\{G(v,v,v), \frac{G(v,\frac{v}{6},\frac{v}{6})(1+G(v,\frac{v}{6},\frac{v}{6})+G(v,\frac{v}{6},\frac{v}{6}))}{1+2G(\frac{v}{6},\frac{v}{6},\frac{v}{6})}\right\} \\ \leq \; & \frac{1}{6}\max\left\{\begin{array}{c} \max\left\{G(v,v,v), \frac{G(v,\frac{v}{6},\frac{v}{6})(1+G(v,\frac{v}{6},\frac{v}{6})+G(v,\frac{v}{6},\frac{v}{6}))}{1+2G(\frac{v}{6},\frac{v}{6},\frac{v}{6})}\right\}, \\ v,v,v \end{array}\right\} \\ \leq \; & \frac{1}{6}\max\{U(v,v,v),\theta(v),\theta(v),\theta(v)\} \\ = \; & \Psi(\alpha(U(v,v,v),\theta(v),\theta(v),\theta(v))) \\ = \; & \Psi(\alpha(U(v,\varrho,\sigma),\theta(v),\theta(\varrho),\theta(\sigma))). \end{aligned}$$

(ii) *If* $v \in [0, \frac{1}{6})$ *and* $\varrho = \sigma \in [\frac{1}{6}, 1]$. *Then,*

$$\begin{aligned}
& \Theta(\alpha(G(\Im v, \Im \varrho, \Im \sigma), \theta(\Im v), \theta(\Im \varrho), \theta(\Im \sigma))) \\
= \; & \max\{G(\Im v, \Im \varrho, \Im \sigma), \theta(\Im v), \theta(\Im v), \theta(\Im v)\} \\
= \; & \max\left\{G\left(\frac{v}{6}, \frac{\varrho}{12}, \frac{\varrho}{12}\right), \frac{v}{6}, \frac{\varrho}{12}, \frac{\varrho}{12}\right\} \\
= \; & \max\left\{\frac{|2v-\varrho|}{6}, \frac{v}{6}, \frac{\varrho}{12}, \frac{\varrho}{12}\right\} \\
\leq \; & \frac{1}{6}\max\left\{\frac{15v+55v\varrho}{9+3|2v-\varrho|}, \frac{v}{6}, \frac{\varrho}{12}, \frac{\varrho}{12}\right\} \\
\leq \; & \frac{1}{6}\max\left\{\max\left\{2|v-\varrho|, \frac{\frac{5v}{3}\left(1+\frac{11\varrho}{6}+\frac{11\varrho}{6}\right)}{1+\frac{|2v-\varrho|}{3}}\right\}, \frac{v}{6}, \frac{\varrho}{12}, \frac{\varrho}{12}\right\} \\
= \; & \frac{1}{6}\max\left\{\begin{array}{c}\max\left\{G(v,\varrho,\varrho), \frac{G(v,\Im v,\Im v)(1+2G(\varrho,\Im \varrho,\Im \varrho))}{1+2G(\Im v,\Im \varrho,\Im \varrho)}\right\}, \\ \theta(v), \theta(\varrho), \theta(\varrho)\end{array}\right\} \\
= \; & \frac{1}{6}\max\left\{\begin{array}{c}\max\left\{G(v,\varrho,\sigma), \frac{G(v,\Im v,\Im v)(1+G(\varrho,\Im \varrho,\Im \varrho)+G(\sigma,\Im \sigma,\Im \sigma))}{1+2G(\Im v,\Im \varrho,\Im \sigma)}\right\}, \\ \theta(v), \theta(\varrho), \theta(\varrho)\end{array}\right\} \\
= \; & \Psi(\alpha(U(v,\varrho,\sigma), \theta(v), \theta(\varrho), \theta(\sigma))).
\end{aligned}$$

(iii) *If* $v \in [\frac{1}{6}, 1]$ *and* $\varrho = \sigma \in [0, \frac{1}{6})$. *Then,*

$$\begin{aligned}
& \Theta(\alpha(G(\Im v, \Im \varrho, \Im \sigma), \theta(\Im v), \theta(\Im \varrho), \theta(\Im \sigma))) \\
= \; & \max\{G(\Im v, \Im \varrho, \Im \sigma), \theta(\Im v), \theta(\Im v), \theta(\Im v)\} \\
= \; & \max\left\{G\left(\frac{v}{12}, \frac{\varrho}{6}, \frac{\varrho}{6}\right), \frac{v}{12}, \frac{\varrho}{6}, \frac{\varrho}{6}\right\} \\
= \; & \max\left\{\frac{|v-2\varrho|}{6}, \frac{v}{12}, \frac{\varrho}{6}, \frac{\varrho}{6}\right\} \\
\leq \; & \frac{1}{6}\max\left\{\frac{66v+220v\varrho}{6+2|2\varrho-v|}, \frac{v}{6}, \frac{\varrho}{12}, \frac{\varrho}{12}\right\} \\
\leq \; & \frac{1}{6}\max\left\{\max\left\{2|v-\varrho|, \frac{\frac{11v}{6}\left(1+\frac{5\varrho}{3}+\frac{5\varrho}{3}\right)}{1+\frac{|2\varrho-v|}{3}}\right\}, \frac{v}{6}, \frac{\varrho}{12}, \frac{\varrho}{12}\right\} \\
= \; & \frac{1}{6}\max\left\{\max\left\{G(v,\varrho,\varrho), \frac{G(v,\Im v,\Im v)(1+2(\varrho,\Im \varrho,\Im \varrho))}{1+2G(\Im v,\Im \varrho,\Im \varrho)}\right\}, \theta(v), \theta(\varrho), \theta(\varrho)\right\} \\
= \; & \Psi(\alpha(U(v,\varrho,\sigma), \theta(v), \theta(\varrho), \theta(\sigma))).
\end{aligned}$$

(iv) *If $\upsilon = \varrho = \sigma \in [\frac{1}{6}, 1]$. Then,*

$$\begin{aligned}
& \Theta(\alpha(G(\Im\upsilon, \Im\varrho, \Im\sigma), \theta(\Im\upsilon), \theta(\Im\varrho), \theta(\Im\sigma))) \\
= \; & \max\{G(\Im\upsilon, \Im\varrho, \Im\sigma), \theta(\Im\upsilon), \theta(\Im\upsilon), \theta(\Im\upsilon)\} \\
= \; & \max\left\{0, \frac{\upsilon}{12}, \frac{\upsilon}{12}, \frac{\upsilon}{12}\right\} = \frac{\upsilon}{12} \leq \frac{1}{6}\left(11\upsilon + \frac{121}{3}\upsilon^2\right) \\
= \; & \frac{1}{6}\max\left\{0, \frac{\frac{11\upsilon}{6}\left(1 + \frac{11\upsilon}{6} + \frac{11\upsilon}{6}\right)}{1+0}\right\} \\
= \; & \frac{1}{6}\max\left\{G(\upsilon, \upsilon, \upsilon), \frac{G\left(\upsilon, \frac{\upsilon}{12}, \frac{\upsilon}{12}\right)\left(1 + G\left(\upsilon, \frac{\upsilon}{12}, \frac{\upsilon}{12}\right) + G\left(\upsilon, \frac{\upsilon}{12}, \frac{\upsilon}{12}\right)\right)}{1 + 2G\left(\frac{\upsilon}{12}, \frac{\upsilon}{12}, \frac{\upsilon}{12}\right)}\right\} \\
\leq \; & \frac{1}{6}\max\left\{\begin{array}{c} \max\left\{G(\upsilon, \upsilon, \upsilon), \frac{G\left(\upsilon, \frac{\upsilon}{6}, \frac{\upsilon}{6}\right)\left(1+G\left(\upsilon, \frac{\upsilon}{6}, \frac{\upsilon}{6}\right)+G\left(\upsilon, \frac{\upsilon}{6}, \frac{\upsilon}{6}\right)\right)}{1+2G\left(\frac{\upsilon}{6}, \frac{\upsilon}{6}, \frac{\upsilon}{6}\right)}\right\}, \\ \upsilon, \upsilon, \upsilon \end{array}\right\} \\
= \; & \frac{1}{6}\max\{U(\upsilon, \upsilon, \upsilon), \theta(\upsilon), \theta(\upsilon), \theta(\upsilon)\} \\
= \; & \Psi(\alpha(U(\upsilon, \upsilon, \upsilon), \theta(\upsilon), \theta(\upsilon), \theta(\upsilon))) \\
= \; & \Psi(\alpha(U(\upsilon, \varrho, \sigma), \theta(\upsilon), \theta(\varrho), \theta(\sigma))).
\end{aligned}$$

Based on the above cases, we conclude that the mapping $\Im$ is a rational $(\Theta, \alpha, \theta, \Psi)$ contraction and fulfills all required of Theorem 4. Hence, it admits a unique FP $\upsilon = 0$ so that $\Im(0) = 0$ and $\theta(0) = 0$.

Corollary 7. *Let (Ω, G) be a complete GMS and $\Im : \Omega \to \Omega$. If $\Im$ fulfills*

$$\alpha(G(\Im\upsilon, \Im\varrho, \Im\sigma), \theta(\Im\upsilon), \theta(\Im\varrho), \theta(\Im\sigma)) > 0,$$

implies

$$\Theta(\alpha(G(\Im\upsilon, \Im\varrho, \Im\sigma), \theta(\Im\upsilon), \theta(\Im\varrho), \theta(\Im\sigma))) \leq \Theta(\alpha(U(\upsilon, \varrho, \sigma), \theta(\upsilon), \theta(\varrho), \theta(\sigma))) - p,$$

for all $\upsilon, \varrho, \sigma \in \Omega$, where $U(\upsilon, \varrho, \sigma)$ is given by Definition 10, $p > 0$, θ is lsc and $\Theta : \mathbb{R}^+ \to \mathbb{R}$ is nondecreasing. Then, $\Im$ is a θ-Picard mapping.

Proof. The results follows immediately if we take $\Psi(\tau) = \Theta(\tau) - p$ in Theorem 4. □

Corollary 8. *Let (Ω, G) be a complete GMS and $\Im : \Omega \to \Omega$. If there is $\delta \in (0, 1)$ so that $\Im$ fulfills $\alpha(G(\Im\upsilon, \Im\varrho, \Im\sigma), \theta(\Im\upsilon), \theta(\Im\varrho), \theta(\Im\sigma)) > 0$ implies*

$$\Theta(\alpha(G(\Im\upsilon, \Im\varrho, \Im\sigma), \theta(\Im\upsilon), \theta(\Im\varrho), \theta(\Im\sigma))) \leq \Theta(\alpha(U(\upsilon, \varrho, \sigma), \theta(\upsilon), \theta(\varrho), \theta(\sigma)))^{\delta},$$

for all $\upsilon, \varrho, \sigma \in \Omega$, where $U(\upsilon, \varrho, \sigma)$ is described as Definition 10, $p > 0$, θ is lsc and $\Theta : \mathbb{R}^+ \to \mathbb{R}$ is nondecreasing. Then, $\Im$ is a θ-Picard mapping.

Proof. We obtain the required by setting $\Psi(\tau) = \Theta(\tau)^{\delta}$ for $\tau > 0$ in Theorem 4. □

Here, the third novel contraction is derived as follows:

Definition 11. *Let (Ω, G) be a GMS. We showed that $\Im : \Omega \to \Omega$ is almost a $(\Theta, \alpha, \theta, \Psi)$ contraction, if there is Θ, Ψ fulfills the conditions $(T_1) - (T_3)$ in Theorem 1 so that for all $\upsilon, \varrho, \sigma \in \Omega$ with $\alpha(G(\Im\upsilon, \Im\varrho, \Im\sigma), \theta(\Im\upsilon), \theta(\Im\varrho), \theta(\Im\sigma)) > 0$, $\Im$ justifies*

$$\begin{aligned}
& \Theta(\alpha(G(\Im\upsilon, \Im\varrho, \Im\sigma), \theta(\Im\upsilon), \theta(\Im\varrho), \theta(\Im\sigma))) \\
\leq \; & \Psi(\alpha(G(\upsilon, \varrho, \sigma) + AV(\upsilon, \varrho, \sigma), \theta(\upsilon), \theta(\varrho), \theta(\sigma))), \qquad (13)
\end{aligned}$$

where

$$V(v,\varrho,\sigma) = \min\{G(v,\Im\varrho,\Im\varrho), G(\varrho,\Im\varrho,\Im\varrho), G(\sigma,\Im\sigma,\Im\sigma)\},$$

and $A > 0$.

Theorem 5. *Let* (Ω, G) *be a complete GMS and* $\Im : \Omega \to \Omega$ *be a rational* $(\Theta, \alpha, \theta, \Psi)$ *contraction mapping. Then,* $\Im$ *is a* θ*-Picard mapping provided that* θ *is lsc.*

Proof. Firstly, we show that $\Lambda(\Im) \subseteq Q_\theta$. Let there be $v \in \Lambda(\Im)$ so that $\theta(v) \neq 0$. Putting $v = \varrho = \sigma$ in (13), we obtain

$$\begin{aligned}
\Theta(\alpha(0,\theta(v),\theta(v),\theta(v))) &= \Theta(\alpha(G(\Im v,\Im v,\Im v),\theta(\Im v),\theta(\Im v),\theta(\Im v))) \\
&\leq \Psi(\alpha(G(v,v,v) + AV(v,v,v),\theta(v),\theta(v),\theta(v))) \\
&= \Psi(\alpha(0,\theta(v),\theta(v),\theta(v))) \\
&< \Theta(\alpha(0,\theta(v),\theta(v),\theta(v))),
\end{aligned}$$

which is a contradiction; hence, $\Lambda(\Im) \subseteq Q_\theta$.

After that, we prove that $\lim_{k\to\infty} G(v_k, v_{k+1}, v_{k+1}) = 0$, $\lim_{k\to\infty} \theta(v_k) = 0$ and $\lim_{k\to\infty} \theta(v_{k+1}) = 0$. Let $v_0 \in \Omega$ and $\{v_k\}$ be a sequence described as $v_k = \Im v_{k-1}$ for all $k \in \mathbb{N}$. If there is $k_0 \in \mathbb{N}$ so that $v_{k_0} = v_{k_0+1}$, that is $v_{k_0} = \Im v_{k_0}$. Hence, v_{k_0} is a FP of $\Im$. Clearly, in this case, $\lim_{k\to\infty} G(v_k, v_{k+1}, v_{k+1}) = 0$, $\lim_{k\to\infty} \theta(v_k) = 0$, $\lim_{k\to\infty} \theta(v_{k+1}) = 0$, and thus, the proof is finished. So, we consider $G(v_k, v_{k+1}, v_{k+1}) > 0$. Letting $v = v_k$, $\varrho = v_{k+1}$ and $\sigma = v_{k+1}$ in (13), we can write

$$\begin{aligned}
&\Theta(\alpha(G(v_{k+1},v_{k+2},v_{k+2}),\theta(v_{k+1}),\theta(v_{k+2}),\theta(v_{k+2}))) \\
=\ &\Theta(\alpha(G(\Im v_k,\Im v_{k+1},\Im v_{k+1}),\theta(\Im v_k),\theta(\Im v_{k+1}),\theta(\Im v_{k+1}))) \\
\leq\ &\Psi(\alpha(G(v_k,v_{k+1},v_{k+1}) + AV(v_k,v_{k+1},v_{k+1}),\theta(v_k),\theta(v_{k+1}),\theta(v_{k+1}))) \\
=\ &\Psi(\alpha(G(v_k,v_{k+1},v_{k+1}),\theta(v_k),\theta(v_{k+1}),\theta(v_{k+1}))),\ (V(v_k,v_{k+1},v_{k+1}) = 0).
\end{aligned}$$

Similarly to the proof of Theorem 3, we have (3). Next, we show that $\{v_k\}$ is a G-Cauchy sequence. If it is not, then by the same method of the proof of Theorem 3, we would obtain (5) and (8). Taking $v = v_{l(j)-1}$, $\varrho = v_{k(j)-1}$ and $\sigma = v_{k(j)-1}$ in (13) and using (T_1), one has

$$\begin{aligned}
&\Theta\Big(\alpha\Big(G\Big(v_{l(j)},v_{k(j)},v_{k(j)}\Big),\theta\Big(v_{l(j)}\Big),\theta\Big(v_{k(j)}\Big),\theta\Big(v_{k(j)}\Big)\Big)\Big) \\
=\ &\Theta\Big(\alpha\Big(G\Big(\Im v_{l(j)-1},\Im v_{k(j)-1},\Im v_{k(j)-1}\Big),\theta\Big(\Im v_{l(j)-1}\Big),\theta\Big(\Im v_{k(j)-1}\Big),\theta\Big(\Im v_{k(j)-1}\Big)\Big)\Big) \\
\leq\ &\Psi\left(\alpha\left(\begin{array}{c} G\Big(v_{l(j)-1},v_{k(j)-1},v_{k(j)-1}\Big) + AV\Big(v_{l(j)-1},v_{k(j)-1},v_{k(j)-1}\Big), \\ \theta\Big(v_{l(j)-1}\Big),\theta\Big(v_{k(j)-1}\Big),\theta\Big(v_{k(j)-1}\Big) \end{array}\right)\right),
\end{aligned}$$

It follows from (3) and (8) that

$$\lim_{j\to\infty}\Big\{G\Big(v_{l(j)-1},v_{k(j)-1},v_{k(j)-1}\Big) + AV\Big(v_{l(j)-1},v_{k(j)-1},v_{k(j)-1}\Big)\Big\} = \epsilon$$

Following the same steps as the proof of Theorem 3, we conclude that $\{v_k\}$ is a G-Cauchy sequence in a complete GMS (Ω, G), and there is $v^* \in \Omega$ so that $\{v_k\}$ is G-convergent to v^*, that is $v_k \to v^*$ as $k \to \infty$ and $\theta(v^*) = 0$.

Now, we prove that $v^* = \Im v^*$. If there is $k_0 \in \mathbb{N}$ so that for $k \geq k_0$, $v_k = \Im v^*$ still holds, then $\Im v^* = \lim_{k\to\infty} v_k = v^*$, that is, $v^* = \Im v^*$. Conversely, put $G(v^*, \Im v^*, \Im v^*) > 0$. Putting $v = v_k$, $\varrho = v^*$ and $\sigma = v^*$ in (13), we have

$$\begin{aligned}
&\Theta(\alpha(G(\Im v_k,\Im v^*,\Im v^*),\theta(\Im v_k),\theta(\Im v^*),\theta(\Im v^*))) \\
\leq\ &\Psi(\alpha(G(v_k,v^*,v^*) + AV(v_k,v^*,v^*),\theta(v_k),0,0)).
\end{aligned}$$

It follows from (T_1) and (T_2) that

$$\begin{aligned} G(v_{k+1},\Im v^*,\Im v^*) &\leq \alpha(G(\Im v_k,\Im v^*,\Im v^*),\theta(\Im v_k),\theta(\Im v^*),\theta(\Im v^*)) \\ &< \alpha(G(v_k,v^*,v^*)+AV(v_k,v^*,v^*),\theta(v_k),0,0), \end{aligned} \tag{14}$$

where

$$\begin{aligned} U(v_k,v^*,v^*) &= \min\{G(v_k,\Im v^*,\Im v^*),G(v^*,\Im v^*,\Im v^*),G(v^*,\Im v^*,\Im v^*)\} \\ &= \min\{G(v_k,\Im v^*,\Im v^*),G(v^*,\Im v^*,\Im v^*)\}\to 0 \text{ as } k\to\infty. \end{aligned}$$

Letting $k\to\infty$ in (14), we obtain

$$G(v^*,\Im v^*,\Im v^*)\leq 0,$$

only this is valid if $v^*=\Im v^*$.

Finally, we illustrate that $v^*=\varrho$, for $v^*,\varrho\in\Lambda(\Im)$. If $v^*\neq\varrho$, then letting $v=v^*$ and $\sigma=\varrho$ in (13), one can write

$$\begin{aligned} &\Theta(\alpha(G(v^*,\varrho,\varrho),0,0,0)) \\ =\ &\Theta(\alpha(G(\Im v^*,\Im\varrho,\Im\varrho),\theta(\Im v^*),\theta(\Im\varrho),\theta(\Im\varrho))) \\ \leq\ &\Psi(\alpha(G(v^*,\varrho,\varrho)+AV(v^*,\varrho,\varrho),\theta(v^*),\theta(\varrho),\theta(\varrho))) \\ =\ &\Psi(\alpha(G(v^*,\varrho,\varrho),0,0,0)) \\ =\ &\Psi(\alpha(G(v^*,\varrho,\varrho),0,0,0)) \\ <\ &\Theta(\alpha(G(v^*,\varrho,\varrho),0,0,0)), \end{aligned}$$

which is a contradiction. So, $v^*=\varrho$. Hence, $\Im$ is a θ-Picard mapping. □

The following example supports Theorem 5:

Example 6. *Assume that $\Omega=[0,1]$ and G is the G-Metric on Ω^3 defined by $G(v,\varrho,\sigma)=|v-\varrho|+|\varrho-\sigma|+|v-\sigma|$, for all $v,\varrho,\sigma\in\Omega$. Obviously, (Ω,G) is a complete GMS. Define the mapping $\Im:\Omega\to\Omega$ by $\Im v=\frac{v}{2}$. If $\Theta(\tau)=\tau$, $\alpha(v,\varrho,\sigma,\ell)=\max\{v,\varrho,\sigma,\ell\}$, $\theta(v)=v$ and $\Psi(\tau)=\frac{\tau}{2}$, for $\tau\in\mathbb{R}^+$. Then, (13) is true. Indeed*

$$\begin{aligned} &\Theta(\alpha(G(\Im v,\Im\varrho,\Im\sigma),\theta(\Im v),\theta(\Im\varrho),\theta(\Im\sigma))) \\ =\ &\alpha(G(\Im v,\Im\varrho,\Im\sigma),\theta(\Im v),\theta(\Im\varrho),\theta(\Im\sigma)) \\ =\ &\max\{G(\Im v,\Im\varrho,\Im\sigma),\Im v,\Im\varrho,\Im\sigma\} \\ =\ &\max\left\{G\left(\frac{v}{2},\frac{\varrho}{2},\frac{\sigma}{2}\right),\frac{v}{2},\frac{\varrho}{2},\frac{\sigma}{2}\right\} \\ =\ &\max\left\{\frac{|v-\varrho|}{2}+\frac{|\varrho-\sigma|}{2}+\frac{|v-\sigma|}{2},\frac{v}{2},\frac{\varrho}{2},\frac{\sigma}{2}\right\} \\ \leq\ &\frac{1}{2}\max\{|v-\varrho|+|\varrho-\sigma|+|v-\sigma|+AV(v,\varrho,\sigma),v,\varrho,\sigma\} \\ \leq\ &\Psi(\alpha(G(v,\varrho,\sigma)+AV(v,\varrho,\sigma),\theta(v),\theta(\varrho),\theta(\sigma))), \end{aligned}$$

Hence, all conditions of Theorem 5 are true. So, 0 is a unique FP of $\Im$ so that $\Im(0)=0$ and $\theta(0)=0$.

Corollary 9. *Let (Ω,G) be a complete GMS and $\Im:\Omega\to\Omega$. If*

$$\alpha(G(\Im v,\Im\varrho,\Im\sigma),\theta(\Im v),\theta(\Im\varrho),\theta(\Im\sigma))>0,$$

implies

$$\begin{aligned} & \Theta(\alpha(G(\Im v, \Im \varrho, \Im \sigma), \theta(\Im v), \theta(\Im \varrho), \theta(\Im \sigma))) \\ \leq\ & \Theta(\alpha(G(v, \varrho, \sigma) + AV(v, \varrho, \sigma), \theta(v), \theta(\varrho), \theta(\sigma))) - p, \end{aligned}$$

for all $v, \varrho, \sigma \in \Omega$, where $V(v, \varrho, \sigma)$ is given by Definition 11 $p > 0$, $A > 0$, θ is lsc and $\Theta : \mathbb{R}^+ \to \mathbb{R}$ is nondecreasing. Then, $\Im$ is a θ-Picard mapping.

Proof. The results follows immediately if we take $\Psi(\tau) = \Theta(\tau) - p$ in Theorem 5. □

Corollary 10. *Let (Ω, G) be a complete GMS and $\Im : \Omega \to \Omega$. If there is $\delta \in (0,1)$ so that $\Im$ fulfills $\alpha(G(\Im v, \Im \varrho, \Im \sigma), \theta(\Im v), \theta(\Im \varrho), \theta(\Im \sigma)) > 0$ implies*

$$\Theta(\alpha(G(\Im v, \Im \varrho, \Im \sigma), \theta(\Im v), \theta(\Im \varrho), \theta(\Im \sigma))) \leq \Theta(\alpha(U(v, \varrho, \sigma), \theta(v), \theta(\varrho), \theta(\sigma)))^{\delta},$$

for all $v, \varrho, \sigma \in \Omega$, where $U(v, \varrho, \sigma)$ is described as Definition 11, $p > 0$, θ lsc and $\Theta : \mathbb{R}^+ \to \mathbb{R}$ is nondecreasing. Then, $\Im$ is a θ-Picard mapping.

Proof. We obtain the required by setting $\Psi(\tau) = \Theta(\tau)^{\delta}$ for $\tau > 0$ in Theorem 5. □

4. θ-Fixed Circle Results

In this part, according to the results of Section 3, we establish some novel θ-fixed disc results in GMS by setting $\alpha(\ell_1, \ell_2, \ell_3, \ell_4) = \ell_1 + \ell_2 + \ell_3 + \ell_4$, or $\alpha(\ell_1, \ell_2, \ell_3, \ell_4) = \max\{\ell_1, \ell_2, \ell_3\} + \ell_4$, or $\alpha(\ell_1, \ell_2, \ell_3, \ell_4) = \max\{\ell_1, \ell_2, \ell_3, \ell_4\}$.

Definition 12. *Let (Ω, G) be a GMS. A mapping $\Im : \Omega \to \Omega$ is said to be a $(\Theta, \theta, \Psi)_{v_0}$-type (I) contraction if there are Θ and Ψ verifying stipulations (T_1) and (T_2) of Theorem 1 so that for each $v \in \Omega$ with $G(\Im v, v, v) + \theta(\Im v) + 2\theta(v) > 0$, $\Im$ satisfies*

$$\Theta(G(\Im v, v, v) + \theta(\Im v) + 2\theta(v)) \leq \Psi(G(v_0, v, v) + \theta(v_0) + 2\theta(v)), \tag{15}$$

where $v_0 \in \Omega$.

Now, by the notion of a circle defined in a metric space, we can generalize Definition 7 as follows:

Definition 13. *Let (Ω, G) be a GMS and $\Im : \Omega \to \Omega$ be a self-mapping and $\theta : \Omega \to [0, \infty)$. Let $r = \inf\{G(\Im v, v, v) : v \neq \Im v\}$. Then, for $v_0 \in \Omega$ and $r \in (0, \infty)$, we say that a circle $\widetilde{C}_G(v_0, r) = \{v \in \Omega : G(v_0, v, v) = r\}$ in Ω is a θ-fixed circle of $\Im$ if $\widetilde{C}_G(v_0, r) \subset \Lambda(\Im) \cap Q_{\theta}$.*

Theorem 6. *Let (Ω, G) be a complete GMS and $\Im : \Omega \to \Omega$ be a $(\Theta, \theta, \Psi)_{v_0}$-type (I) contraction with the point v_0 and the number r described as in Definition 13. If $\theta(v_0) = G(\Im v_0, v_0, , v_0) = 0$ and $\theta(v) \leq G(\Im v, v, v)$ for each $v \in \widetilde{C}_G(v_0, r)$; then, $\widetilde{C}_G(v_0, r)$ is a θ-fixed circle of $\Im$.*

Proof. For all $v \in \widetilde{C}_G(v_0, r)$, we show that $v = \Im v$. If $r = 0$. $\widetilde{C}_G(v_0, r) = \{v_0\}$, then, clearly $\widetilde{C}_G(v_0, r)$ is θ-fixed circle of $\Im$. So, assume that $r > 0$ and $v \neq \Im v$ for all $v \in \widetilde{C}_G(v_0, r)$. Using (15), the definition of r, $\theta(v_0) = 0$, and $v_0 = \Im v_0$, we have

$$\begin{aligned} \Theta(G(\Im v, v, v) + \theta(\Im v) + 2\theta(v)) &\leq \Psi(G(v_0, v, v) + \theta(v_0) + 2\theta(v)) \\ &= \Psi(G(v_0, v, v) + 2\theta(v)) \\ &< \Theta(G(v_0, v, v) + 2\theta(v)) \\ &\leq \Theta(r + 2\theta(v)) \\ &\leq \Theta(G(\Im v, v, v) + 2\theta(v)). \end{aligned}$$

Applying the condition (T_2), we obtain

$$G(\Im v, v, v) + \theta(\Im v) + 2\theta(v) < G(\Im v, v, v) + 2\theta(v),$$

which is a contradiction. So, $v \neq \Im v$. As $\theta(v) \leq G(v, \Im v, \Im v)$ for each $v \in \widetilde{C}_G(v_0, v)$, we obtain that

$$\theta(v) \leq G(v, \Im v, \Im v) = 0.$$

Hence, $\theta(v) = 0$, that is, $\widetilde{C}_G(v_0, r)$ is a θ-fixed circle of $\Im$. $\square$

Remark 4. *Theorem 6 is true if we replace $G(v_0, v, v)$ with one of the following:*

(i)

$$G_1(v_0, v, v) = \max\left\{ \begin{array}{c} G(v_0, v, v), G(\Im v, v, v), G(\Im v_0, v_0, v_0), \\ \frac{G(\Im v_0, v, v) + G(\Im v, v_0, v_0)}{2} \end{array} \right\};$$

(ii)

$$\begin{aligned} G_2(v_0, v, v) \quad &= \quad c_1 G(v_0, v, v) + c_2 G(\Im v, v, v) + c_3 G(\Im v_0, v_0, v_0) \\ &\quad + c_4 G(\Im v_0, v, v) + c_5 G(\Im v, v_0, v_0), \end{aligned}$$

where $c_j \in [0, 1)$ with $\sum_{j=1}^{5} c_j = 1$;

(iii)

$$G_1(v_0, v, v) = \max\left\{ \begin{array}{c} G(v_0, v, v), \alpha G(\Im v, v, v), (1-\alpha) G(\Im v_0, v_0, v_0), \\ (1-\alpha) G(\Im v, v, v), \alpha G(\Im v_0, v_0, v_0), \frac{G(\Im v_0, v, v) + G(\Im v, v_0, v_0)}{2} \end{array} \right\},$$

where $v \in \Omega$, $\alpha \in [0, 1]$. In addition, if we replaced the condition $G(\Im v_0, v_0, , v_0) = 0$ with $G(\Im v_0, v_0, , v_0) \leq r$ for each $v \in \widetilde{C}_G(v_0, r)$, then the findings are still valid.

Example 7. *Let $\Omega = \{-6, -2, 0, 1, 2, 5\}$ and the mapping $G : \Omega^3 \to \Omega$ be defined by $G(v, \varrho, \sigma) = |v - \varrho| + |\varrho - \sigma| + |v - \sigma|$, for all $v, \varrho, \sigma \in \Omega$. Clearly, (Ω, G) is a complete GMS. Define $\Im : \Omega \to \Omega$ and $\theta : \Omega \to [0, \infty)$ by*

$$\Im v = \left\{ \begin{array}{ll} v, & \text{if } v \neq 5, \\ 1, & \text{otherwise,} \end{array} \right.$$

and

$$\theta(v) = \left\{ \begin{array}{cc} 0, & \text{if } v \in \{-6, 1, 5\}, \\ v^3 - 4v, & \text{otherwise,} \end{array} \right.$$

respectively. Then, we obtain $r = 8$, $\Lambda(\Im) = \Omega - \{5\}$, $Q_\theta = \Omega$ and $\Lambda(\Im) \cap Q_\theta = \Omega - \{5\}$. Now, we illustrate that $\Im$ is a $(\Theta, \theta, \Psi)_{v_0}$-type (I) contraction with $v_0 = -2$, $\Theta(\tau) = \tau$, and $\Psi(\tau) = \frac{3}{4}\tau$. Indeed, if $v = 5$, we obtain that

$$\begin{aligned} G(\Im 5, 5, 5) + \theta(\Im 5) + 2\theta(5) \quad &= \quad 8 < \frac{21}{2} = \frac{3}{4} \times 14 \\ &= \quad \frac{3}{4} G(-2, 5, 5) + \theta(-2) + 2\theta(5). \end{aligned}$$

Therefore, all conditions of Theorem 6 and Corollary 11 are fulfilled by $\Im$. Hence, $\widetilde{C}_G(-2, 8) = \{2\}$ is a θ-fixed circle of $\Im$.

Corollary 11. *Let (Ω, G) be a complete GMS and $\Im : \Omega \to \Omega$ be a self-mapping. If there are continuous Θ and Ψ satisfying (T_1) of Theorem 1 and so that $\Im$ verifies*

$$G(\Im v, v, v) > 0 \text{ implies } \Theta(G(\Im v, v, v)) \leq \Psi(G(v_0, v, v)),$$

and $G(\Im v_0, v_0,, v_0) = 0$*, where* $v_0 \in \Omega$*. If* r *is defined by Definition 13. Then, the circle* $\widetilde{C}_G(v_0, r)$ *is a* θ*-fixed circle of* $\Im$*.*

Proof. In Theorem 6, set $\theta(\tau) = 0$. □

Corollary 12. *Let* (Ω, G) *be a complete GMS and the number* r *is defined by Definition 13. If there is* $p > 0$ *and a nondecreasing function* $\Theta : \mathbb{R}^+ \to \mathbb{R}$ *so that* $\Im : \Omega \to \Omega$ *verifies*

$$\Theta(G(\Im v, v, v) + \theta(\Im v) + 2\theta(v)) \leq \Theta(G(v_0, v, v) + \theta(v_0) + 2\theta(v)) - p,$$

for all $v, v_0 \in \Omega$ *with* $G(\Im v, v, v) + \theta(\Im v) + 2\theta(v) > 0$, $\theta(v_0) = G(\Im v_0, v_0,, v_0) = 0$*, and* $\theta(v) \leq G(\Im v, v, v)$ *for each* $v \in \widetilde{C}_G(v_0, r)$*. Then, the circle* $\widetilde{C}_G(v_0, r)$ *is a* θ*-fixed circle of* $\Im$*.*

Proof. Put $\Psi(\tau) = \Theta(\tau) - \tau$ in Theorem 6. □

Corollary 13. *Let* (Ω, G) *be a complete GMS and the number* r *is defined by Definition 13. If there is* $\delta \in [0, 1)$ *and nondecreasing function* $\Theta : \mathbb{R}^+ \to \mathbb{R}$ *so that* $\Im : \Omega \to \Omega$ *fulfills*

$$\Theta(G(\Im v, v, v) + \theta(\Im v) + 2\theta(v)) \leq \Theta(G(v_0, v, v) + \theta(v_0) + 2\theta(v))^{\delta},$$

for all $v, v_0 \in \Omega$ *with* $G(\Im v, v, v) + \theta(\Im v) + 2\theta(v) > 0$, $\theta(v_0) = G(\Im v_0, v_0,, v_0) = 0$*, and* $\theta(v) \leq G(\Im v, v, v)$ *for each* $v \in \widetilde{C}_G(v_0, r)$*. Then, the circle* $\widetilde{C}_G(v_0, r)$ *is a* θ*-fixed circle of* $\Im$*.*

Proof. Setting $\Psi(\tau) = \Theta(\tau)^{\delta} - \tau$ in Theorem 6. □

Corollary 14. *Let* (Ω, G) *be a complete GMS and the number* r *is defined by Definition 13. If there is* $\delta \in [0, 1)$ *and a nondecreasing function* $\Theta : \mathbb{R}^+ \to \mathbb{R}$ *so that* $\Im : \Omega \to \Omega$ *fulfills*

$$\Theta(G(\Im v, v, v)) \leq \Theta(G(v_0, v, v))^{\delta},$$

for all $v, v_0 \in \Omega$ *with* $G(\Im v, v, v) > 0$ *and* $G(\Im v_0, v_0,, v_0) = 0$*, then, the circle* $\widetilde{C}_G(v_0, r)$ *is a* θ*-fixed circle of* $\Im$*.*

Proof. To obtain the result, we take $\theta(\tau) = 0$ in Corollary 13. □

Definition 14. *Let* (Ω, G) *be a GMS. A mapping* $\Im : \Omega \to \Omega$ *is said to be a* $(\Theta, \theta, \Psi)_{v_0}$*-type* (II) *contraction if there are* Θ *and* Ψ *satisfying stipulations* (T_1) *and* (T_2) *of Theorem 1 so that* $\Im$ *fulfills*

$$\Theta(\max\{G(\Im v, v, v), \theta(v)\} + \theta(\Im v)) \leq \Psi(\max\{G(v_0, v, v), \theta(v)\} + \theta(v_0)), \qquad (16)$$

for each $v \in \Omega$ *with* $\max\{G(\Im v, v, v), \theta(v)\} + \theta(\Im v) > 0$*, where* $v_0 \in \Omega$*.*

Theorem 7. *Let* (Ω, G) *be a complete GMS and* $\Im : \Omega \to \Omega$ *be a* $(\Theta, \theta, \Psi)_{v_0}$*-type* (II) *contraction with the point* v_0 *and the number* r *described as in Definition 13. If* $\theta(v_0) = G(\Im v_0, v_0,, v_0) = 0$ *and* $\theta(v) \leq G(\Im v, v, v)$ *for each* $v \in \widetilde{C}_G(v_0, r)$*, then* $\widetilde{D}_G(v_0, r)$ *is a* θ*-fixed circle of* $\Im$*.*

Proof. For all $v \in \widetilde{C}_G(v_0, r)$, we show that $v = \Im v$. If $r = 0$. $\widetilde{C}_G(v_0, r) = \{v_0\}$, then, it is easy to see that $\widetilde{C}_G(v_0, r)$ is a θ-fixed circle of $\Im$. So, assume that $r > 0$ and $v \neq \Im v$ for all $v \in \widetilde{C}_G(v_0, r)$. By (16), the definition of r, $\theta(v_0) = 0$, and $v_0 = \Im v_0$, one has

$$\begin{aligned}
\Theta(\max\{G(\Im v, v, v), \theta(v)\} + \theta(\Im v)) &\leq \Psi(\max\{G(v_0, v, v), \theta(v)\} + \theta(v_0)) \\
&= \Psi(\max\{G(v_0, v, v), \theta(v)\}) \\
&< \Theta(\max\{r, \theta(v)\}) \\
&\leq \Theta(\max\{G(\Im v, v, v), \theta(v)\})
\end{aligned}$$

It follows from (T_2) that

$$\max\{G(\Im v, v, v), \theta(v)\} + \theta(\Im v) < \max\{G(\Im v, v, v), \theta(v)\},$$

which is a contradiction. So, $v \neq \Im v$. Since $\theta(v) \leq G(v, \Im v, \Im v)$ for each $v \in \widetilde{C}_G(v_0, v)$, we obtain

$$\theta(v) \leq G(v, \Im v, \Im v) = 0.$$

Hence, $\theta(v) = 0$, that is, $\widetilde{C}_G(v_0, r)$ is a θ-fixed circle of $\Im$. □

Remark 5. *If we replace $G(v_0, v, v)$ with the same conditions (i), or (ii), or (iii) presented in Remark 4, Theorem 6 remains valid.*

Example 8. *Let $\Omega = \{-6, -3, -2, 0, 1, 2, 3, 5\}$ be equipped with $G(v, \varrho, \sigma) = |v - \varrho| + |\varrho - \sigma| + |v - \sigma|$. It is easy to see that (Ω, G) is a complete GMS. Define the self-mapping $\Im : \Omega \to \Omega$ and the function $\theta : \Omega \to [0, \infty)$ by*

$$\Im v = \begin{cases} v, & \text{if } v \neq 5, \\ 2, & \text{otherwise}, \end{cases}$$

and

$$\theta(v) = \begin{cases} 0, & \text{if } v \in \{-6, 1, 5\}, \\ v^5 - 13v^3 + 36v, & \text{otherwise}, \end{cases}$$

respectively. Then, we have $r = 6$, $\Lambda(\Im) = \Omega - \{5\}$, $Q_\theta = \Omega$ and $\Lambda(\Im) \cap Q_\theta = \Omega - \{5\}$. Now, we prove that $\Im$ is a $(\Theta, \theta, \Psi)_{v_0}$-type (II) contraction with $v_0 = 0$, $\Theta(\tau) = \tau$, and $\Psi(\tau) = \frac{3}{4}\tau$. Indeed, if $v = 5$, we can write

$$\begin{aligned} \max\{G(\Im 5, 5, 5), \theta(5)\} + \theta(\Im 5) &= 6 < \frac{25}{3} = \frac{5}{6} \times 10 \\ &= \frac{5}{6} \max\{G(0, 5, 5), \theta(5)\} + \theta(0). \end{aligned}$$

Therefore, all conditions of Theorem 7 are fulfilled by $\Im$. Hence, $\widetilde{C}_G(0, 6) = \{3, -3\}$ is a θ-fixed circle of $\Im$.

Corollary 15. *Let (Ω, G) be a complete GMS and the number r is defined by Definition 13. If there is $p > 0$ and nondecreasing function $\Theta : \mathbb{R}^+ \to \mathbb{R}$ so that $\Im : \Omega \to \Omega$ satisfies*

$$\Theta(\max\{G(\Im v, v, v), \theta(v)\} + \theta(\Im v)) \leq \Theta(\max\{G(v_0, v, v) + \theta(v)\} + \theta(v_0)) - p,$$

for all $v, v_0 \in \Omega$ with $\max\{G(\Im v, v, v), \theta(v)\} + \theta(\Im v) > 0$, $\theta(v_0) = G(\Im v_0, v_0, , v_0) = 0$, and $\theta(v) \leq G(\Im v, v, v)$ for each $v \in \widetilde{C}_G(v_0, r)$. Then, the circle $\widetilde{C}_G(v_0, r)$ is a θ-fixed circle of $\Im$.

Proof. Put $\Psi(\tau) = \Theta(\tau) - \tau$ in Theorem 7. □

Corollary 16. *Let (Ω, G) be a complete GMS and the number r is defined by Definition 13. If there is $\delta \in [0, 1)$ and a nondecreasing function $\Theta : \mathbb{R}^+ \to \mathbb{R}$ so that $\Im : \Omega \to \Omega$ fulfills*

$$\Theta(\max\{G(\Im v, v, v), \theta(v)\} + \theta(\Im v)) \leq \Theta(\max\{G(v_0, v, v) + \theta(v)\} + \theta(v_0))^{\delta},$$

for all $v, v_0 \in \Omega$ with $\max\{G(\Im v, v, v), \theta(v)\} + \theta(\Im v) > 0$, $\theta(v_0) = G(\Im v_0, v_0, , v_0) = 0$, and $\theta(v) \leq G(\Im v, v, v)$ for each $v \in \widetilde{C}_G(v_0, r)$. Then, the circle $\widetilde{C}_G(v_0, r)$ is a θ-Fixed circle of $\Im$.

Proof. Setting $\Psi(\tau) = \Theta(\tau)^{\delta} - \tau$ in Theorem 7. □

Definition 15. *Let (Ω, G) be a GMS. A mapping $\Im : \Omega \to \Omega$ is called a $(\Theta, \theta, \Psi)_{v_0}$-type (III) contraction if there are Θ and Ψ satisfying stipulations (T_1) and (T_2) of Theorem 1 so that*

$$\Theta(\max\{G(\Im v, v, v), \theta(v), \theta(\Im v)\}) \leq \Psi(\max\{G(v_0, v, v), \theta(v), \theta(v_0)\}), \tag{17}$$

for each $v \in \Omega$ with $\max\{G(\Im v, v, v), \theta(v), \theta(\Im v)\} > 0$, where $v_0 \in \Omega$.

Theorem 8. *Let (Ω, G) be a complete GMS and $\Im : \Omega \to \Omega$ be a $(\Theta, \theta, \Psi)_{v_0}$-type (III) contraction with the point v_0 and the number r described as in Definition 13. If $\theta(v_0) = G(\Im v_0, v_0, , v_0) = 0$ and $\theta(v) \leq G(\Im v, v, v)$ for each $v \in \widetilde{C}_G(v_0, r)$, then $\widetilde{D}_G(v_0, r)$ is a θ-fixed circle of $\Im$.*

Proof. For all $v \in \widetilde{C}_G(v_0, r)$, we show that $v = \Im v$. If $r = 0$. $\widetilde{C}_G(v_0, r) = \{v_0\}$, then, it is easy to see that $\widetilde{C}_G(v_0, r)$ is a θ-fixed circle of $\Im$. So, assume that $r > 0$ and $v \neq \Im v$ for all $v \in \widetilde{C}_G(v_0, r)$. By (17), the definition of r, $\theta(v_0) = 0$, and $v_0 = \Im v_0$, one obtains

$$\begin{aligned}
\Theta(\max\{G(\Im v, v, v), \theta(v), \theta(\Im v)\}) &\leq \Psi(\max\{G(v_0, v, v), \theta(v), \theta(v_0)\}) \\
&= \Psi(\max\{G(v_0, v, v), \theta(v)\}) \\
&< \Theta(\max\{G(v_0, v, v), \theta(v)\}) \\
&= \Theta(\max\{r, \theta(v)\}) \\
&\leq \Theta(\max\{G(\Im v, v, v), \theta(v)\}) \\
&\leq \Theta(G(\Im v, v, v)).
\end{aligned}$$

By the properties of Θ, we have

$$\max\{G(\Im v, v, v), \theta(v)\} \leq \max\{G(\Im v, v, v), \theta(v), \theta(\Im v)\} < G(\Im v, v, v),$$

which is a contradiction. Hence, $v \neq \Im v$. Because $\theta(v) \leq G(v, \Im v, \Im v)$ for each $v \in \widetilde{C}_G(v_0, v)$, we have

$$\theta(v) \leq G(v, \Im v, \Im v) = 0.$$

Therefore, $\theta(v) = 0$, that is, $\widetilde{C}_G(v_0, r)$ is a θ-fixed circle of $\Im$. □

Remark 6. *If we replace $G(v_0, v, v)$ with the same conditions (i), or (ii), or (iii) presented in Remark 4, Theorem 8 remains true.*

Example 9. *If we take the same assumptions of Examples 7 and 8, we can find that the requirements of Theorem 8 are satisfied by $\Im$. Hence, $\widetilde{C}_G(0, 6) = \{3, -3\}$ and is a θ-fixed circle of $\Im$.*

Remark 7. *According to Example 9, we note that a θ-fixed circle of $\Im$ is not unique.*

Corollary 17. *Let (Ω, G) be a complete GMS and the number r is defined by Definition 13. If there is $p > 0$ and nondecreasing function $\Theta : \mathbb{R}^+ \to \mathbb{R}$ so that $\Im : \Omega \to \Omega$ satisfies*

$$\Theta(\max\{G(\Im v, v, v), \theta(v), \theta(\Im v)\}) \leq \Theta(\max\{G(v_0, v, v) + \theta(v), \theta(v_0)\}) - p,$$

for all $v, v_0 \in \Omega$ with $\max\{G(\Im v, v, v), \theta(v), \theta(\Im v)\} > 0$, $\theta(v_0) = G(\Im v_0, v_0, , v_0) = 0$, and $\theta(v) \leq G(\Im v, v, v)$ for each $v \in \widetilde{C}_G(v_0, r)$. Then, the circle $\widetilde{C}_G(v_0, r)$ is a θ-fixed circle of $\Im$.

Proof. Put $\Psi(\tau) = \Theta(\tau) - \tau$ in Theorem 8. □

Corollary 18. *Let (Ω, G) be a complete GMS and the number r is defined by Definition 13. If there is $\delta \in [0, 1)$ and nondecreasing function $\Theta : \mathbb{R}^+ \to \mathbb{R}$ so that $\Im : \Omega \to \Omega$ fulfills*

$$\Theta(\max\{G(\Im v, v, v), \theta(v), \theta(\Im v)) \leq \Theta(\max\{G(v_0, v, v) + \theta(v), \theta(v_0)\})^{\delta},$$

for all $\upsilon, \upsilon_0 \in \Omega$ *with* $\max\{G(\upsilon_0, \upsilon, \upsilon) + \theta(\upsilon), \theta(\upsilon_0)\} > 0, \theta(\upsilon_0) = G(\Im\upsilon_0, \upsilon_0, , \upsilon_0) = 0$, *and* $\theta(\upsilon) \leq G(\Im\upsilon, \upsilon, \upsilon)$ *for each* $\upsilon \in \widetilde{C}_G(\upsilon_0, r)$. *Then, the circle* $\widetilde{C}_G(\upsilon_0, r)$ *is a* θ*-fixed circle of* $\Im$.

Proof. Setting $\Psi(\tau) = \Theta(\tau)^{\delta} - \tau$ in Theorem 8. □

5. Supportive Application

In this part, the results of Theorem 3 are applied to find the existence of solution for the nonlinear integral below:

$$\upsilon(r) = \varkappa(r) + \int_{\sigma}^{\rho} \mho(r, \varsigma, \upsilon(\varsigma))d\varsigma, \tag{18}$$

where $\sigma, \rho \in \mathbb{R}$ with $\sigma \leq \rho$, $\upsilon \in C[\sigma, \rho]$ (the set of all real continuous functions on the closed interval $[\sigma, \rho]$), $\varkappa : [\sigma, \rho] \to \mathbb{R}$, and $\mho : [\sigma, \rho] \times [\sigma, \rho] \times \mathbb{R} \to \mathbb{R}$ are continuous functions. Assume that $\Omega = C[\sigma, \rho]$ endowed with the standard G-Metric

$$G_\infty(\upsilon, \varrho, \ell) = \sup_{r\in[\sigma,\rho]} |\upsilon(r) - \varrho(r)| + \sup_{r\in[\sigma,\rho]} |\varrho(r) - \ell(r)| + \sup_{r\in[\sigma,\rho]} |\ell(r) - \upsilon(r)|,$$

for all $\upsilon, \varrho, \sigma \in \Omega$. It is obvious that (Ω, G) is a G-complete. Define the mapping $\Im : \Omega \to \Omega$ by

$$\Im\upsilon(r) = \varkappa(r) + \int_{\sigma}^{\rho} \mho(r, \varsigma, \upsilon(\varsigma))d\varsigma, \text{ for } r \in [\sigma, \rho]. \tag{19}$$

It is obvious that the solution of Equation (18) corresponds to the FP of $\Im$ in Equation (19). Following that, we shall demonstrate our finding.

Theorem 9. *The integral Equation (18) has a solution if there exists* $p > 0$ *such that*

$$|\mho(r, \varsigma, \upsilon(\varsigma)) - \mho(r, \varsigma, \varrho(\varsigma))| \leq \frac{1}{\rho - \sigma}|\upsilon - \varrho|,$$

for $\upsilon, \varrho \in \Omega$ *and* $r \in [\sigma, \rho]$.

Proof. Describe the control functions $\alpha = \alpha_1$, $\theta(\tau) = 0$ and $\Psi(\tau) = \Theta(\tau)$, for $\tau \in \Omega$, where $\Theta : \mathbb{R}^+ \to \mathbb{R}$ is nondecreasing function. Then

$$\begin{aligned} & |\Im\upsilon - \Im\varrho| \\ = & \left| \int_{\sigma}^{\rho} \mho(r, \varsigma, \upsilon(\varsigma))d\varsigma - \mho(r, \varsigma, \varrho(\varsigma))d\varsigma \right| \\ \leq & \int_{\sigma}^{\rho} |\mho(r, \varsigma, \upsilon(\varsigma)) - \mho(r, \varsigma, \varrho(\varsigma))| d\varsigma \\ \leq & \frac{1}{\rho - \sigma} \int_{\sigma}^{\rho} |\upsilon - \varrho| d\varsigma \\ = & |\upsilon - \sigma|. \end{aligned} \tag{20}$$

Similarly,

$$|\Im\varrho - \Im\ell| \leq |\varrho - \ell|, \tag{21}$$

and

$$|\Im\ell - \Im\upsilon| \leq |\ell - \upsilon|. \tag{22}$$

Summing (20) to (22) and taking the suprimum, we have

$$\begin{aligned} G_\infty(\Im v, \Im \varrho, \Im \ell) &= \sup_{r\in[\sigma,\rho]} |\Im v - \Im \varrho| + \sup_{r\in[\sigma,\rho]} |\Im \varrho - \Im \ell| + \sup_{r\in[\sigma,\rho]} |\Im \ell - \Im v| \\ &\le \sup_{r\in[\sigma,\rho]} |v - \sigma| + \sup_{r\in[\sigma,\rho]} |\varrho - \ell| + \sup_{r\in[\sigma,\rho]} |\ell - v| \\ &= G_\infty(v, \varrho, \ell). \end{aligned}$$

Taking Θ in both sides, one has

$$\Theta(G_\infty(\Im v, \Im \varrho, \Im \ell)) \le \Theta(G_\infty(v, \varrho, \ell)),$$

which implies that

$$\Theta(\alpha(G(\Im v, \Im \varrho, \Im \ell), \theta(\Im v), \theta(\Im \varrho), \theta(\Im \ell))) \le \Psi(\alpha(G(v, \varrho, \ell), \theta(v), \theta(\varrho), \theta(\ell))).$$

It is evident that $\Im$ meets the requirements of Theorem 3. Hence, $\Im$ has an FP, which means that the integral Equation (18) has a solution. □

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